DIFFERENTIABILITY, SUMMABILITY, AND FIXED POINTS IN BANACH SPACES

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This document consists of three main chapters. Each chapter considers a topic within functional analysis.

The first chapter focuses on fixed point theory. Our main result in this chapter is to show the existence of a fixed point free contractive map on a weakly compact and convex set. This answers a long-standing open question. We also prove a theorem about transfinite iterates of contractive maps on weakly compact sets converging to a fixed point.

The second chapter concerns summability theory. We prove a theorem quantifying the extent to which iterated Cesàro averaging can (and cannot) bring a divergent sequence closer to convergence. We develop a method which generates Banach limits which are invariant under certain operators which generate summability methods, including the Cesàro method. We also develop a constructive method for defining transfinite iterates of a "translated" Cesàro operator corresponding to certain limit ordinals. These iterates usually have non-constructive definitions.

In the third chapter we consider a couple unusual reformulations of the derivative in Banach spaces and sees what becomes of the theory of differentiation. We show that through the use of "difference transforms" one can re-work much of the foundation of Banach space differential calculus. Sometimes this re-working leads to a much more efficient development of the theory. Sometimes the results generated are different from their ordinary Fréchet derivative counterparts. And sometimes, as is the case with the Inverse Function Theorem, the results and proofs are necessarily very similar to their known versions. Our main theorem

is to show that in the presence of certain geometric conditions relating to smoothness, these difference transforms can be made to vary continuously in a way that is more consistent with their behavior in Hilbert spaces. We also clarify Henri Cartan's use of the term "strong derivative".

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Preface

I dedicate these pages to my wife Amanda and our daughter Elena, who spent much of the beginning of her life watching her parents type their theses.

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I. FIXED POINT THEORY

Given a set S and a function $F : S \to S$, a **fixed point** of F is an element $x \in S$ such that $F(x) = x$. Fixed point theory is concerned with the following question. What conditions can we put on the set S and the function F to guarantee the existence of a fixed point for F?

Two particularly famous results of this type are as follows.

Theorem I.1. ***[Brouwer]*** *Let $K \subseteq \mathbb{R}^n$ be non-empty, compact, and convex. Let $f : K \to K$. If f is continuous, then f has a fixed point.*

Theorem I.2. ***[Schauder]*** *Let X be a Banach space and let $K \subseteq X$ be non-empty, convex, and compact. If $f : K \to K$ is continuous then f has a fixed point.*

This property of K, that every continous function on it has a fixed point, is sometimes called the **topological fixed point property**. The version of Schauder's Theorem above can be summarized as "compact, convex sets in Banach spaces have the topological fixed point property". The primary interest of this chapter is not the topological fixed point property. However, it is worth noting that in 2001 this result was generalized by Cauty ([12]) to metric linear spaces that are not necessarily locally convex.

There are limits to the topological fixed point property. Notice that the domains in the results above are closed, bounded, and convex (sometimes abbreviated c.b.c.). This pattern of only considering c.b.c. sets when attempting to guarantee a fixed point property will hold for the rest of this document. To see why, consider the following examples.

Example I.3. *Let D_1 be the interval $(0, 1)$. Consider $f : D_1 \to D_1$ given by $f(x) = x/2$. This function is fixed point free. Notice that D_1 is bounded and convex but not closed.*

Example I.4. *Let $D_2 = [0, \infty)$. Consider $g : D_2 \to D_2$ given by $g(x) = x + 1$. g is fixed point free. Notice that D_2 is closed and convex but not bounded.*

Example I.5. *Let $D_3 \subseteq \mathbb{R}^2$ be the unit circle. That is, let $D_3 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $h : D_3 \to D_3$ be rotation by some non-zero angle, i.e. fix $\theta_0 \in (0, 2\pi)$ and let $h((\cos(\theta), \sin(\theta))) = (\cos(\theta + \theta_0), \sin(\theta + \theta_0))$. This map is fixed point free. Notice that D_3 is closed and bounded but not convex.*

So we restrict our attention to c.b.c. sets. And since we are interested in Banach spaces, we also turn our attention away from the topological fixed point property because of the following result.

Theorem I.6. ***[Klee,** [32]]* *Let X be a Banach space and let $C \subseteq X$ be c.b.c. and non-compact. There exists a continuous function $f : C \to C$ that lacks a fixed point.*

So, in order to guarantee the existence of a fixed point for a class of functions, we must consider a smaller class than that of all continuous functions. The type of restrictive conditions on functions that we will consider are called metric conditions. Perhaps the best-known such result is Banach's contraction mapping theorem which is as follows.

Definition I.7. *Let (M, d) be a metric space. A map $f : M \to M$ is a* ***strict contraction*** *if there is some $k \in (0, 1)$ such that $d(f(x), f(y)) \leq k\, d(x, y)$ for every $x, y \in M$.*

Theorem I.8. ***[Banach]*** *Let (M, d) be a complete metric space (such as a closed subset of a Banach space). Let $f : M \to M$ be a strict contraction. Then f has a unique fixed point.*

At this point we pause to affirm the utility of such results outside of their value as answers to natural questions. For example, here is a problem from a graduate preliminary examination from [15, p. 58, Su84].

Question I.9. *Show that there is a unique continuous function $f : [0, 1] \to \mathbb{R}$ such that*

$$f(x) = \sin(x) + \int_0^1 \frac{f(y)}{e^{x+y+1}} dy.$$

Proof. Consider $G : (\mathcal{C}[0, 1], \|\cdot\|_\infty) \to (\mathcal{C}[0, 1], \|\cdot\|_\infty)$ given by

$$G(f)(x) = \sin(x) + \int_0^1 \frac{f(y)}{e^{x+y+1}} dy.$$

The result is shown if we can show that G has a unique fixed point. We will get this fact from Theorem I.8 once we show that G is a contraction.

$$\begin{aligned}
\|Gf - Gg\|_\infty &= \sup_{x\in[0,1]} \left| \int_0^1 \frac{f(y)}{e^{x+y+1}} dy - \int_0^1 \frac{g(y)}{e^{x+y+1}} dy \right| \\
&\leq \int_0^1 \frac{|f(y)-g(y)|}{e^{0+y+1}} dy \leq \|f-g\|_\infty \int_0^1 e^{-y-1}\, dy = (e^{-1}-e^{-2})\|f-g\|_\infty.
\end{aligned}$$

The claim is proven because $e^{-1} - e^{-2}$ is between 0 and 1. □

This example is a simple exercise. There are applications of fixed point theory to differential and integral equations beyond such simple examples. In [27, ch. 13] the authors point us to more involved applications and a number of references.

Turning our attention back to the theory, note that Theorem I.8 raises the following question. Is there a way to weaken the condition from strict contraction to something similar and maintain the guarantee of a fixed point? We already know from examples I.4 and I.5 that weakening the metric condition to **isometric** (f is isometric if $d(f(x), f(y)) = d(x, y)$ for all x, y) leads to a negative answer. But what about in the presence of a c.b.c. domain?

It turns out that, when considering non-compact sets, a metric condition weaker than strict contraction will not generally guarantee a fixed point. We will need to add some some geometric conditions in order to guarantee a fixed point for all members of some particular classes of continuous functions. But before that, we need one more result (beyond Klee's Theorem above) to narrow our focus onto the right kinds of continuous functions. What follows is not exactly the way the original authors phrased their result. However, people studying the minimal displacement problem have noted that their result is even stronger than what is stated below (see e.g. [27, ch. 19]).

Definition I.10. *Given a metric space (M, d) and a real number $k > 0$, a function $f : M \to M$ is said to be **k-Lipschitz** if $d(f(x), f(y)) \leq k\, d(x, y)$ for all $x, y \in M$.*

Theorem I.11. ***[Lin and Sternfeld, [36]]*** *Suppose X is a Banach space and $C \subseteq X$ is c.b.c. and non-compact. Let $k > 1$ be given. There exists a k-Lipschitz map $T : C \to C$ such that T is fixed point free.*

In the presence of these results (Banach Contraction and Lin-Sternfeld), the fixed point question only remains interesting for k-Lipschitz maps when $k = 1$. We use the following terms.

Definition I.12. *A map $T : C \to C$ is **nonexpansive** if $\|T(x) - T(y)\| \leq \|x - y\|$ $\forall x, y \in C$.*

Definition I.13. *A map $T : C \to C$ is **contractive** if $\|T(x) - T(y)\| < \|x - y\|$ for all $x, y \in C$ such that $x \neq y$.*

To see that it is possible to have a fixed point free contractive map on a c.b.c. set, consider the following example, which is borrowed from page 12 of [27].

Example I.14. *Consider the Banach space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, which is the space of continuous real-valued function on $[0,1]$ equipped with the norm of uniform convergence. Let K be defined as*

$$K = \{f \in \mathcal{C}[0,1] : 0 = f(0) \leq f(t) \leq f(1) = 1\}.$$

K is closed, bounded, and convex. Define $T : K \to K$ by $Tf(t) = t\,f(t)$. T defined in this way is contractive and fixed point free.

Such examples can be found in many c.b.c. sets in many spaces. However, there are certain geometric conditions that can be put on the space X or the c.b.c. set K that will guarantee the existence of a fixed point for every [isometric, contractive, or nonexpansive] map. One of the best-known results of this type is due to Art Kirk and is as follows. We will define some terms from the theorem after stating it.

Theorem I.15. ***Kirk*** *Let $(X, \|\cdot\|)$ be a Banach space. Let $K \subseteq X$ be weakly compact (which implies closed and bounded) and convex. Suppose further that K has normal structure. Then every nonexpansive $T : K \to K$ has a fixed point.*

Before we define normal structure, we need a couple of geometric definitions.

Definition I.16. *Let $(X, \|\cdot\|)$ be a Banach space. Let $D \subseteq X$ be any non-empty set. For any $x \in D$, define **the radius of D about x** to be $r_x(D) = \sup\{\|x - y\| : y \in D\}$. Then we define **the radius of D** to be the quantity $rad(D) = \inf\{r_x(D) : x \in D\}$ and **the diameter of D** to be the quantity $diam(D) = \sup\{r_x(D) : x \in D\}$.*

Definition I.17. *Given X and D as in the previous definition, a point $x \in D$ is said to be* ***diametral*** *if $r_x(D) = diam(D)$.*

D is said to have ***normal structure*** *if every bounded convex $S \subseteq D$ with $diam(S) > 0$ contains a nondiametral point; or equivalently, $rad(S) < diam(S)$.*

We will not discuss normal structure further. On the other hand, we will discuss weakly compact sets extensively in this chapter. We will assume the usual definition of the dual space $(X^*, \|\cdot\|_{X^*})$ of a given Banach space $(X, \|\cdot\|)$ over the scalar field $\mathbb{K}$.

Definition I.18. *For every $\varphi \in X^*$ and every open $U \subseteq \mathbb{K}$, $\varphi^{-1}(U)$ is a sub-basic open set in the* ***weak topology*** *on X. From this sub-basis, we define the weak topology to be the collection of all arbitrary unions of finite intersections of sub-basic open sets.*

Given this definition of the weak topology, we say that a set is weakly compact if it is compact in the weak topology. For a more thorough discussion of the weak topology see [48, ch. 3].

From 1965, when [31] was published, it was unknown if the normal structure condition could be dropped from Theorem I.15. In 1981, Dale Alspach [3] provided an example of a fixed point free isometry of a weakly compact and convex set. We will present a slight alteration of Alspach's example. The weakly compact and convex we will consider is

$$C_{1/2} = \left\{ f \in L^1[0,1] : 0 \le f(t) \le 1 \text{ for all } t \text{ and } \int_0^1 f = \frac{1}{2} \right\}.$$

Alspach's map, $T : C_{1/2} \to C_{1/2}$ is given by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 1 & : 0 \le t < 1/2, \\ (2f(2t-1) - 1) \vee 0 & : 1/2 \le t \le 1. \end{cases}$$

Here, for all $\alpha, \beta \in \mathbb{R}$, $\alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$.

Again, Alspach's map is an isometry. That is, $\|Tf - Tg\|_1 = \|f - g\|_1$ for all $f, g \in C_{1/2}$. What was unknown, until the main result of this chapter, was whether it was possible to replace Alspach's map with a contractive map and get the same result. That is, the answer to the following question was unknown.

Question I.19. *Is there a weakly compact and convex set C in a Banach space and a contractive map $R : C \to C$ such that R is fixed point free?*

It is straightforward to see that there are functions which contract the distances between some pairs of points. Taking T to be Alspach's map and $I : C_{1/2} \to C_{1/2}$ to be the identity, define $F : C_{1/2} \to C_{1/2}$ to be $F = \frac{I+T}{2}$. F indeed maps into $C_{1/2}$ because of the convexity of that set. F is fixed point free because T is fixed point free (if $F(f) = f$, then $\frac{f+Tf}{2} = f$, implying that $\frac{Tf}{2} = \frac{f}{2}$ and therefore $Tf = f$). Also, F is the average of nonexpansive maps and is therefore nonexpansive.

To see that F contracts some pairs of point consider $f_1 = \chi_{[0,1/2]}$ and $f_2 = \chi_{[1/2,1]}$. $\|f_1 - f_2\|_1 = 1$. $Tf_1 = \chi_{[0,1/4]} + \chi_{[1/2,3/4]}$ and $Tf_2 = \chi_{[1/4,1/2]} + \chi_{[3/4,1]}$. So

$$F(f_1) = \chi_{[0,1/4]} + \frac{1}{2}\chi_{[1/4,1/2]} + \frac{1}{2}\chi_{[1/2,3/4]}$$

and

$$F(f_2) = \frac{1}{2}\chi_{[1/4,1/2]} + \frac{1}{2}\chi_{[1/2,3/4]} + \chi_{[3/4,1]}.$$

Then, noting that $F(f_1) - F(f_2) = \chi_{[0,1/4]} - \chi_{[3/4,1]}$, we see $\|F(f_1) - F(f_2)\| = \frac{1}{2}$.

But F does not contract the distance between every pair of unequal points in $C_{1/2}$. Consider, for example, $u = \chi_{[0,1/4)} + \frac{1}{2}\chi_{[1/4,3/4)}$ and $v = \frac{1}{2}\chi_{[0,1]}$. One can check that $\|u - v\|_1 = \|Fu - Fv\|_1 = 1/4$. However, the function $\frac{I+T^2}{2}$ does contract the pair u, v, and this begins to suggest our solution to I.19.

First we note that contractive mappings on c.b.c. sets that are not weakly compact arise quite often. For example, Maurey [41] showed that every weakly compact convex subset C in the Banach space c_0 of scalar sequences that converge to zero, with the usual $\|\cdot\|_\infty$-norm, is such that every nonexpansive map $U : C \to C$ has a fixed point. On the other hand, Dowling, Lennard and Turett [21] showed the following converse result: on every non-weakly compact, closed bounded convex set C in $(c_0, \|\cdot\|_\infty)$, there exists a nonexpansive mapping $W : C \to C$ that is fixed point free. This W can be arranged to be contractive.

Finally, our answer to question I.19 is as follows.

Theorem I.20. *The mapping*

$$R : C_{1/2} \to C_{1/2} : f \mapsto \sum_{n=0}^{\infty} \frac{T^n f}{2^{n+1}} = \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \cdots \right)(f)$$

is contractive and fixed point free on $C_{1/2}$.

A. PROOF OF THE MAIN THEOREM

In this section we will prove Theorem I.20. This proof has been accepted for publication in roughly this form [8]. All sets that are the domains of a mapping are assumed to be non-empty. For this section and the next, T will stand for Alspach's map as defined above.

First, let us confirm that R maps $C_{1/2}$ back into $C_{1/2}$. Fix an arbitrary $f \in C_{1/2}$. For each n we have $0 \le T^n f \le 1$, therefore $0 \le Rf \le 1$. Further,

$$\int_0^1 Rf\, dm = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 T^n f\, dm = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 f\, dm = \frac{1}{2}\ .$$

We will begin the proof that $R : C_{1/2} \to C_{1/2}$ is contractive and fixed point free by defining for every $f \in C_{1/2}$ the set

$$A_n(f) = \{x \in [0,1] : T^n f(x) \in (0,1)\}.$$

It turns out, as we make precise in the following lemma, that $\lim_{n\to\infty} m(A_n) = 0$.

Lemma I.21. *For every* $f \in C_{1/2}$, $m(A_n(f)) \le 2^{-n}$.

Proof. In what follows we will ignore certain dyadic numbers in the domain. These constitute a set of measure zero.

Let $f \in C_{1/2}$ b given. Decompose the set

$$A_1(f) = (A_1(f) \cap [0,1/2)) \cup (A_1(f) \cap (1/2,1]).$$

If $x \in A_1(f) \cap [0,1/2)$, then $x \in [0,1/2)$ and $Tf(x) \in (0,1)$. By definition, for $x \in [0,1/2)$, $Tf(x) = 2f(2x) \wedge 1$. So

$$x \in [0,1/2) \text{ and } f(2x) \in (0,1/2) \quad \Leftrightarrow \quad x \in A_1(f) \cap [0,1/2).$$

Similarly, if $x \in A_1(f) \cap (1/2, 1]$, then $x \in (1/2, 1]$ and $Tf(x) \in (0, 1)$. By definition, for $x \in (1/2, 1]$, $Tf(x) = (2f(2x-1) - 1) \vee 0$. So

$$x \in (1/2, 1] \text{ and } f(2x-1) \in (1/2, 1) \quad \Leftrightarrow \quad x \in A_1(f) \cap (1/2, 1].$$

Note that

$$m\{x \in [0, 1/2) : f(2x) \in (0, 1/2)\} = \frac{1}{2} m\{x \in (0, 1) : f(x) \in (0, 1/2)\}$$

and

$$m\{x \in (1/2, 1] : f(2x-1) \in (1/2, 1)\} = \frac{1}{2} m\{x \in (0, 1) : f(x) \in (1/2, 1)\}.$$

Putting this together gives

$$\begin{aligned}
\frac{1}{2} m(A_0(f)) &= \frac{1}{2} m[f \in (0, 1/2)] + \frac{1}{2} m[f = 1/2] + \frac{1}{2} m[f \in (1/2, 1)] \\
&\geq \frac{1}{2} m[f \in (0, 1/2)] + \frac{1}{2} m[f \in (1/2, 1)] \\
&= m(A_1(f) \cap (1/2, 1]) + m(A_1(f) \cap [0, 1/2)) \\
&= m(A_1(f)).
\end{aligned}$$

We have shown that $m(A_1(f)) \leq \frac{1}{2} m(A_0(f))$ for any $f \in C_{1/2}$. Now we can apply this result to $T^{n-1}f$ and use the fact that $A_{j+k}(f) = A_j(T^k f)$ to get

$$m(A_n(f)) = m(A_1(T^{n-1}f)) \leq \frac{1}{2} m(A_0(T^{n-1}f)) = \frac{1}{2} m(A_{n-1}(f)),$$

giving $m(A_n(f)) \leq \dfrac{1}{2^n} m(A_0(f)) \leq \dfrac{1}{2^n} \to 0$ for any $f \in C_{1/2}$. $\square$

Lemma I.22. *Let $h \in C_{1/2}$, and let y be any non-dyadic number in $[0, 1]$. Also, let $n \in \mathbb{N}$. If $h(y) = 0$, then for all $j \in \{1, 2, ..., 2^n\}$*

$$T^n h\left(\frac{y + j - 1}{2^n}\right) = 0.$$

If $h(y) = 1$, then for all $j \in \{1, 2, ..., 2^n\}$

$$T^n h\left(\frac{y + j - 1}{2^n}\right) = 1.$$

Proof: We will prove the lemma by induction on n. Beginning with $n = 1$, we need to check that the claim holds for $j \in \{1, 2\}$.

Since $y/2$ is between 0 and $1/2$, we see that $Th(y/2) = 2h(y) \wedge 1$, which is 1 when $h(y) = 1$ and zero when $h(y) = 0$. Thus the claim holds for $j = 1$.

As $(y+1)/2$ is between $1/2$ and 1, we have $Th((y+1)/2) = (2h(y)-1) \vee 0$, which agrees with h when h is either 1 or 0. Hence the claim holds when $j = 2$.

Inductively, suppose for all $j \in \{1, 2, ..., 2^m\}$ that when $h(y)$ is 0 or 1,

$$h(y) = T^m h\left(\frac{y+j-1}{2^m}\right).$$

Applying the base case to $T^m h$ and $k \in \{1, 2\}$ we have that for all $j \in \{1, 2, ..., 2^m\}$,

$$T^m h\left(\frac{y+j-1}{2^m}\right) = T^{m+1} h\left(\frac{\frac{y+j-1}{2^m} + k - 1}{2}\right)$$

It follows from this fact and the inductive assumption that

$$\begin{aligned} h(y) &= T^{m+1} h\left(\frac{\frac{y+j-1}{2^m} + k - 1}{2}\right) \\ &= T^{m+1} h\left(\frac{y+j+2^m(k-1)-1}{2^{m+1}}\right). \end{aligned}$$

When $k = 1$ we have $j + 2^m(k-1) = j$ spanning $\{1, 2, ..., 2^m\}$. When $k = 2$ we have $j + 2^m(k-1) = j + 2^m$ spanning $\{2^m+1, 2^m+2, ..., 2^{m+1}\}$. $\square$

Lemma I.23. *For every f and g in $C_{1/2}$ with $\|f-g\|_1 > 0$ there is some $N \in \mathbb{N}$ such that*

$$\left\|\frac{I+T^N}{2}f - \frac{I+T^N}{2}g\right\|_1 < \|f-g\|_1.$$

Proof: Fix $f, g \in C_{1/2}$. Note that all sets in the domain can vary up to a set of measure zero without affecting the argument. Define the sets

$$B_n = \{x \in [0,1] : T^n f(x) \in (0,1) \text{ or } T^n g(x) \in (0,1)\} = A_n(f) \cup A_n(g),$$

$$C_n = \{x \in [0,1] : T^n f(x) = 1 \text{ and } T^n g(x) = 0\},$$

$$D_n = \{x \in [0,1] : (T^n f(x) = 1 \text{ and } T^n g(x) = 1) \text{ or }$$
$$(T^n f(x) = 0 \text{ and } T^n g(x) = 0)\},$$

$$E_n = \{x \in [0,1] : T^n f(x) = 0 \text{ and } T^n g(x) = 1\}.$$

Note that $[0,1] = B_n \cup C_n \cup D_n \cup E_n$ is a disjoint union.

By Lemma I.21 we have $m(B_n) \to 0$. Because $\|f - g\|_1 > 0$ and $\int_0^1 f = \int_0^1 g = 1/2$, it follows that $m[f > g] > 0$ and $m[g > f] > 0$. Here $[f > g] = \{x : f(x) > g(x)\}$. This notation is common, and we will use it throughout the rest of the proof.

Now we will check that there is some N_0 so that when $n > N_0$ we have $m(C_n) > 0$ and $m(E_n) > 0$. Note that

$$\begin{aligned}
\|f - g\|_1 &= \|T^n f - T^n g\|_1 \\
&= \int_{B_n} |T^n f - T^n g| + \int_{D_n} 0 + \int_{C_n} 1 + \int_{E_n} 1 \\
&= \int_{B_n} |T^n f - T^n g| + m(C_n) + m(E_n).
\end{aligned}$$

This gives $m(E_n) + m(C_n) = \|f - g\|_1 - \int_{B_n} |T^n f - T^n g|$. Also, $\int T^n f = \int T^n g = \frac{1}{2}$ which gives

$$\int_{B_n} (T^n f - T^n g) + \int_{C_n} (T^n f - T^n g) + \int_{D_n} (T^n f - T^n g) + \int_{E_n} (T^n f - T^n g) = 0$$

$$\Rightarrow \int_{B_n} (T^n f - T^n g) + \int_{C_n} 1 + \int_{E_n} (-1) = 0$$

$$\Rightarrow m(E_n) - m(C_n) = \int_{B_n} (T^n f - T^n g).$$

We know that $|T^n f(x) - T^n g(x)| \leq 1$, and so we can deduce from these facts that

$$\|f - g\|_1 \geq m(E_n) + m(C_n) \geq \|f - g\|_1 - m(B_n)$$

and

$$|m(E_n) - m(C_n)| \leq m(B_n).$$

Now, from the fact that $m(B_n) \to 0$ it follows that
$m(E_n) \to \frac{1}{2}\|f-g\|_1$ and $m(C_n) \to \frac{1}{2}\|f-g\|_1$.

So, we choose n to be sufficiently large so that $m(E_n)$ and $m(C_n)$ are both greater than $\frac{1}{4}\|f-g\|_1$. By Lemma I.22 we have that for all $k \in \mathbb{N}$,

$$C_{n+k} \supseteq \bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} C_n \right) \text{ and } E_{n+k} \supseteq \bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} E_n \right).$$

We now make the following claim.

(♠) Claim. There exists $k \in \mathbb{N}$ such that

$$S_1 := E_{n+k} \cap [f > g] \text{ and } S_2 := C_{n+k} \cap [f < g]$$

both have positive measure.

[Proof of (♠)] Let $W := [f > g]$. Fix $\varepsilon > 0$. By, for example Royden [46], Chapter 3, Proposition 15, there exists a finite sequence of open intervals $(I_l)_{l=1}^{\nu}$ such that for $\Gamma := \cup_{l=1}^{\nu} I_l$, $m(W \,\Delta\, \Gamma) < \varepsilon$. Without loss of generality, we may assume $(I_l)_{l=1}^{\nu}$ is pairwise disjoint, and that each interval I_l is a dyadic interval of the form $(j_l/2^k, (j_l+1)/2^k)$, for some $j_l \in \{0, \ldots, 2^k - 1\}$, and some $k \in \mathbb{N}$. We may write

$$\chi_\Gamma = \sum_{j=0}^{2^k-1} \beta_j \, \chi_{(j/2^k,(j+1)/2^k)} \ ,$$

where each $\beta_j \in \{0,1\}$. Then

$$\begin{aligned}
m(E_{n+k} \cap W) &\geq m\left(\bigcup_{j=0}^{2^k-1}\left(\frac{j}{2^k}+\frac{1}{2^k}E_n\right)\cap W \cap \Gamma\right) \\
&\geq m\left(\bigcup_{j=0}^{2^k-1}\left(\frac{j}{2^k}+\frac{1}{2^k}E_n\right)\cap \Gamma\right) - m\left(\bigcup_{j=0}^{2^k-1}\left(\frac{j}{2^k}+\frac{1}{2^k}E_n\right)\cap \Gamma\backslash W\right) \\
&\geq \int_0^1 \sum_{j=0}^{2^k-1} \chi_{\left(\frac{j}{2^k}+\frac{1}{2^k}E_n\right)} \sum_{s=0}^{2^k-1} \beta_s \, \chi_{\left(\frac{s}{2^k},\frac{s+1}{2^k}\right)} \, dm - m\left(\Gamma\backslash W\right) \\
&> \int_0^1 \sum_{j=0}^{2^k-1} \beta_j \, \chi_{\left(\frac{j}{2^k}+\frac{1}{2^k}E_n\right)} \, dm - \varepsilon = m(E_n)\,\frac{1}{2^k}\sum_{j=0}^{2^k-1}\beta_j - \varepsilon \\
&= m(E_n)\, m(\Gamma) - \varepsilon > m(E_n)\,\big(m(W)-\varepsilon\big) - \varepsilon \geq m(E_n)\, m(W) - 2\,\varepsilon \\
&\geq \frac{\|f-g\|_1}{4}\, m(W) - 2\,\varepsilon > \frac{\|f-g\|_1}{8}\, m(W) > 0 \;,
\end{aligned}$$

for $\varepsilon \in (0,\infty)$ chosen small enough.

Similarly, there exists $k_2 \in \mathbb{N}$ such that we also have

$$m(C_{n+k_2} \cap [f<g]) > \frac{\|f-g\|_1}{4}\, m([f<g]) - 2\,\varepsilon > \frac{\|f-g\|_1}{8}\, m[f<g] > 0$$

for an even smaller choice of $\varepsilon \in (0,\infty)$. Moreover, from above we see that we may choose k and k_2 to be equal.

[End proof of (♠)]

Finally, letting $N = n+k$ we can compute the cancellation. Let $S_3 = [0,1]\backslash(S_1 \cup S_2)$.

$$\begin{aligned}
\left\|\frac{I+T^N}{2}f - \frac{I+T^N}{2}g\right\|_1 &= \int_0^1 \left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\left|\frac{f-g-1}{2}\right| + \int_{S_2}\left|\frac{f+1-g}{2}\right| + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\frac{1+g-f}{2} + \int_{S_2}\frac{1+f-g}{2} + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&< \int_{S_1}\frac{1+f-g}{2} + \int_{S_2}\frac{1+g-f}{2} + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\left(\left|\frac{T^Nf-T^Ng}{2}\right| + \left|\frac{f-g}{2}\right|\right) + \int_{S_2}\left(\left|\frac{T^Nf-T^Ng}{2}\right| + \left|\frac{f-g}{2}\right|\right) \\
&\qquad + \int_{S_3}\left|\frac{f-g}{2} + \frac{T^Nf-T^Ng}{2}\right|
\end{aligned}$$

$$\leq \int_0^1 \left|\frac{f-g}{2}\right| + \left|\frac{T^N f - T^N g}{2}\right| = \|f-g\|_1. \quad \square$$

Corollary I.24. *R is contractive. That is, for all f and g in $C_{1/2}$ with $\|f-g\|_1 > 0$ we have*

$$\|Rf - Rg\|_1 < \|f-g\|_1.$$

Proof: This follows from Lemma I.23 and the fact that we can re-write R in the following way:

$$\begin{aligned} Rf &= \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \frac{T^3}{16} + \cdots\right) f \\ &= \frac{1}{2}\frac{I+T}{2} f + \frac{1}{4}\frac{I+T^2}{2} f + \frac{1}{8}\frac{I+T^3}{2} f + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{I+T^n}{2} f. \end{aligned}$$

Each of the pieces $\frac{I+T^n}{2}$ is nonexpansive. Lemma I.23 gives that every pair $f \neq g$ is contracted by at least one piece, and therefore it is contracted by R. $\square$

Before the final lemma, we need yet one more reformulation of R.

$$\begin{aligned} Rf &= \frac{f}{2} + \frac{Tf}{4} + \frac{T^2 f}{8} + \frac{T^3 f}{16} + \cdots \\ &= \frac{f}{2} + \frac{1}{2}\left(\frac{Tf}{2} + \frac{T(Tf)}{4} + \frac{T^2(Tf)}{8} + \cdots\right) \\ &= \frac{I}{2} f + \frac{1}{2} R(T(f)) = \frac{I+RT}{2} f. \end{aligned}$$

Lemma I.25. *R is fixed point free on $C_{1/2}$.*

Proof: Because R is contractive and $T : C_{1/2} \to C_{1/2}$ is an isometry we have that for all $f, g \in C_{1/2}$ with $\|f-g\|_1 > 0$

$$\|RTf - RTg\|_1 < \|Tf - Tg\|_1 = \|f-g\|_1.$$

But then,

$$\begin{aligned} \|Rf - Rg\|_1 &= \left\|\frac{f-g}{2} + \frac{RTf - RTg}{2}\right\|_1 \\ &\geq \left\|\frac{f-g}{2}\right\|_1 - \left\|\frac{RTf - RTg}{2}\right\|_1 \\ &> 0. \end{aligned}$$

This gives that R is 1-1 on $C_{1/2}$ as a subset of L^1. Now let f_0 be any fixed point of R in this set. We have

$$\begin{aligned} f_0 &= \frac{f_0}{2} + \frac{Tf_0}{4} + \frac{T^2 f_0}{8} + \frac{T^3 f_0}{16} + \cdots \\ \Rightarrow \frac{f_0}{2} &= \frac{Tf_0}{4} + \frac{T^2 f_0}{8} + \frac{T^3 f_0}{16} + \cdots \\ \Rightarrow f_0 &= \frac{Tf_0}{2} + \frac{T^2 f_0}{4} + \frac{T^3 f_0}{8} + \cdots = R(Tf_0). \end{aligned}$$

But then $R(f_0) = R(Tf_0)$, with R 1-1, implies $Tf_0 = f_0$, giving a fixed point of Alspach's map in $C_{1/2}$. This is known to be impossible ([3]). □

We have now finished the proof of Theorem I.20. By Lemmas I.24 and I.25, R is contractive and fixed point free.

B. OTHER CONTRACTIVE AND FIXED POINT FREE MAPS

Now that we know of one fixed point free contractive self-map of a weakly compact convex set, we can ask questions about how common such maps are. We might consider a map such as Δ from [19] given by

$$\Delta f(t) = \begin{cases} f(2t) & : 0 \leq t < 1/2, \\ 1 - f(2t-1) & : 1/2 \leq t < 1. \end{cases}$$

In [19] it was shown that the map $T\Delta : C_{1/2} \to C_{1/2}$ was nonexpansive and fixed point free. In light of our main theorem above, the following question then becomes natural.

Question I.26. *Is* $\displaystyle\sum_{n=0}^{\infty} \frac{(T\Delta)^n}{2^{n+1}}$ *contractive and fixed point free?*

It turns out that the answer to this question is yes. The details are presented in a later section. But first, let's address the following more general questions.

Question I.27. *Given some nonexpansive* $F : C_{1/2} \to C_{1/2}$, *is* $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ *contractive?*

Question I.28. *Given some nonexpansive* $F : C_{1/2} \to C_{1/2}$, *is* $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ *fixed point free?*

Letting $F = I$, Theorem I.20 tells us that the answer to questions I.27 and I.28 is at least "sometimes yes". Let us see that the answer is also "sometimes no". To do this, we will use the fact that Alspach's map is invertible in a sense. We will also see that Alspach's map is not completely invertible. Define $G : C_{1/2} \to C_{1/2}$ by

$$Gf(t) = \frac{1}{2}f\left(\frac{t}{2}\right) + \frac{1}{2}f\left(\frac{1}{2} + \frac{t}{2}\right).$$

Now we can check that

$$\begin{aligned} GTf(t) &= \frac{1}{2}Tf\left(\frac{t}{2}\right) + \frac{1}{2}Tf\left(\frac{1}{2} + \frac{t}{2}\right) \\ &= \frac{1}{2}(2f(t) \wedge 1) + \frac{1}{2}((2f(t) - 1) \vee 0) \\ &= \left(f(t) \wedge \frac{1}{2}\right) + \left(\left(f(t) - \frac{1}{2}\right) \vee 0\right). \end{aligned}$$

If $f(t) \in [0, 1/2)$, this is $f(t) + 0 = f(t)$. If $f(t) \in [1/2, 1]$, this is $\frac{1}{2} + (f(t) - \frac{1}{2}) = f(t)$.

This means G is a left inverse for T. However, G is not a right inverse for T. For $t \in [0, 1/2)$ we can compute.

$$\begin{aligned} TGf(t) &= 2Gf(2t) \wedge 1 = 2\left(\frac{1}{2}f(t) + \frac{1}{2}f\left(\frac{1}{2} + t\right)\right) \wedge 1 \\ &= \left(f(t) + f\left(\frac{1}{2} + t\right)\right) \wedge 1. \end{aligned}$$

This seems to indicate that $TGf(t) > f(t)$ is possible (or even likely) for $t \in [0, 1/2)$. This is a feature of Alspach's map, and has been called **left heaviness**. We will explore this further after checking that T does indeed lack a right inverse.

To see that $T : C_{1/2} \to C_{1/2}$ lacks a right inverse, consider the function $f_{1/2} \in C_{1/2}$ (again, this is $f_{1/2} = \frac{1}{2}\chi_{[0,1]}$) and let $H : C_{1/2} \to C_{1/2}$ be any function. Suppose that $THf_{1/2} = f_{1/2}$. We then know by inspecting $t \in [0, 1/2)$ that $(T(Hf_{1/2}))(t) = 1/2$, implying that $Hf_{1/2}(t) = 1/4$ for all $t \in [0, 1]$. But then this implies that for every $t \in (1/2, 1]$ we have $T(Hf_{1/2})(t) = 0$ which is not $1/2 = f_{1/2}(t)$.

Yet we can use G to show that the answer to questions I.27 and I.28 is sometimes "no". Note that

$$\sum_{n=0}^{\infty} \frac{(TG)^n}{2^{n+1}} = \frac{I}{2} + \frac{TG}{4} + \frac{(TG)^2}{8} + \frac{(TG)^3}{16} \cdots$$

and because

$$(TG)^n = (TG)(TG)\cdots(TG)(TG) = T(GT)(G\cdots T)(GT)G = TG$$

we can re-write this as

$$\sum_{n=0}^{\infty} \frac{(TG)^n}{2^{n+1}} = \frac{I}{2} + \frac{TG}{2}.$$

This gives that any fixed point of TG will be a fixed point of $\sum \frac{(TG)^n}{2^{n+1}}$. Now consider $g = \chi_{[0,1/2]} \in C_{1/2}$.

$$\begin{aligned} G(g)(t) &= \frac{1}{2} g\left(\frac{t}{2}\right) + \frac{1}{2} g\left(\frac{1}{2} + \frac{t}{2}\right) \\ &= \frac{1}{2}\chi_{[0,1]}(t) + \frac{1}{2} \cdot 0 = \frac{1}{2}\chi_{[0,1]}(t) = f_{1/2}(t). \end{aligned}$$

Since $T(f_{1/2}) = \chi_{[0,1/2]}$, we have that $TG(\chi_{[0,1/2]}) = \chi_{[0,1/2]}$ and this is a fixed point of $\sum \frac{(TG)^n}{2^{n+1}}$. This tells us that the answer to question I.28 is sometimes "no".

If we can find one more fixed point, then we will have that this function is also not contractive. To this end, consider $h = \chi_{[0,1/4]} + \chi_{[1/2,3/4]} \in C_{1/2}$.

$$\begin{aligned} G(h)(t) &= \frac{1}{2} h\left(\frac{t}{2}\right) + \frac{1}{2} h\left(\frac{1}{2} + \frac{t}{2}\right) \\ &= \frac{1}{2}\chi_{[0,1/2]}(t) + \frac{1}{2}\chi_{[0,1/2]}(t) = \chi_{[0,1/2]}. \end{aligned}$$

So $TGh = T(\chi_{[0,1/2]}) = \chi_{[0,1/4]} + \chi_{[1/2,3/4]} = h$, and $h \neq g$ is a second fixed point of $\sum \frac{(TG)^n}{2^{n+1}}$. This says $\sum \frac{(TG)^n}{2^{n+1}}$ is not contractive on $C_{1/2}$ (in fact it is not even contractive on D_∞, the minimal invariant set for T, which is described a little later). This says the answer to question I.27 is sometimes "no". Now we re-phrase those questions as classification questions.

Open Question I.29. *For which nonexpansive $F : C_{1/2} \to C_{1/2}$, is $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ contractive?*

Open Question I.30. *For which nonexpansive $F : C_{1/2} \to C_{1/2}$, is $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ fixed point free?*

It may be helpful in these kinds of considerations to pass to a smaller domain that still has the nice properties of $C_{1/2}$. Here, we may consider D_∞, which is the unique minimal invariant set for Alspach's map on $C_{1/2}$ ([14]). It is known that D_∞ is a weakly compact and convex subset of $C_{1/2}$, and $T : D_\infty \to D_\infty$ is fixed point free. Some other properties of D_∞ have been published. The following Lemmas, which come from internal discussions ([7]), may offer some additional insight into D_∞.

In what follows, $f_{1/2} = \frac{1}{2}\chi_{[0,1]}$ is the weak limit of the iterates of Alspach's map starting with any element of $C_{1/2}$. The sets $D_n = D_n(f_{1/2})$ are the sets involved in the construction of the minimal invariant set $D_\infty \subseteq C_{1/2}$ as found in [14].

To facilitate reading the proofs of these Lemmas, we note that $D_0 = \{f_{1/2}\}$, $D_n = conv\{D_{n-1} \cup T(D_{n-1})\}$, and $D_\infty = \overline{\cup_{n=0}^{\infty} D_n}$.

Lemma I.31. *For any $n \in \mathbb{N}$, every $f \in D_n(f_{1/2})$ is such that*

$$1 = f(x) + f(1-x) \text{ for all } x \neq \frac{k}{2^n}, \text{ for all } k \in \{0, 1, ..., 2^n\}.$$

Proof. We will proceed by induction. For the base case we must check the function $f_{1/2}$. But we know $f_{1/2}(x) = 1/2$ for all x. So

$$1 = \frac{1}{2} + \frac{1}{2} = f_{1/2}(x) + f_{1/2}(1-x).$$

Now suppose $n \in \mathbb{N}$ is fixed and the claim holds for all $f \in D_{n-1}$. We need to check that the condition holds for Tf.

To this end, take any $x \neq k/2^n$. Without loss of generality, suppose $x < 1/2$. Note that neither $2x$ nor $1 - 2x$ is of the form $j/2^{n-1}$ for any integer j.

With the assumption $x < 1/2$ we can compute

$$Tf(x) = 2f(2x) \wedge 1 \text{ and } Tf(1-x) = (2f(1-2x) - 1) \vee 0.$$

Set $f(2x) = a$. Our inductive assumption gives $f(1 - 2x) = 1 - a$. Now we check the quantity

$$Tf(x) + Tf(1-x) = (2a \wedge 1) + ((2(1-a) - 1) \vee 0)$$
$$= (2a \wedge 1) + ((1 - 2a) \vee 0).$$

If $2a \geq 1$ this is $1 + 0 = 1$. Otherwise $2a < 1$ and this quantity is $2a + (1 - 2a) = 1$. $\square$

Lemma I.32. *Suppose $g \in D_\infty$. Then $g(x) + g(1-x) = 1$ except on a set of measure zero.*

It has been pointed out that there is a straightforward proof of this Lemma using the topology of $C_{1/2}$. The following elementary proof is included because it is also relatively simple.

Proof. Suppose, to get a contradiction, that there is a $g \in D_\infty$ and $\epsilon, \delta \in (0,1)$ such that

$$\mu([|g(x) + g(1-x) - 1| > \delta]) > \epsilon.$$

Then let $f \in D_n$ for some $n \in \mathbb{N}$ with $\|f - g\|_1 < \frac{1}{4}\delta\epsilon$.

Define the set

$$B = \{x \in [0,1] : [|f(x) - g(x)| \geq \delta/2] \text{ or } [|f(1-x) - g(1-x)| \geq \delta/2]\}.$$

The fact that $\|f - g\|_1 < \frac{1}{4}\delta\epsilon$ tells us that $\mu(B) < \epsilon$.

But then for all $x \in B^c$ with x not a dyadic rational we have by the previous Lemma that

$$\begin{aligned} |1 - (g(x) + g(1-x))| &= |1 - (f(x) + f(1-x)) + (f(x) + f(1-x)) - (g(x) + g(1-x))| \\ &\leq |f(x) - g(x)| + |f(1-x) + g(1-x)| < \delta. \end{aligned}$$

This contradicts the first assumption of the proof. □

These kinds of considerations relating to minimal invariant sets are important throughout fixed point theory. As such, we note that R defined above, which was found to be contractive and fixed point free, also has the property that $R : D_\infty \to D_\infty$. This follows immediately from the fact that D_∞ is closed and convex and $R(f) = \sum \frac{T^n f}{2^{n+1}}$ is an infinite convex combination of elements of D_∞. This means lemmas like Lemma I.32 and what follows will pass to a minimal invariant set for R.

Open Question I.33. *What is the minimal invariant set for R?*

Lemma I.34. *Every $f \in D_n$ is such that for every integer k between 0 and $2^{n-1} - 1$*

$$f(x) \geq f\left(x + \frac{1}{2^n}\right) \text{ for all } x \in \left(\frac{2k}{2^n}, \frac{2k+1}{2^n}\right).$$

Proof. We will proceed by induction, beginning with $n = 1$. Let $f \in D_1$. We know that f must be a convex combination of the constantly 1/2 seed function $f_{1/2}$ and the function $T(f_{1/2}) = \chi_{[0,1/2]}$.

We know that this function is of the form (for some $t \in (0, 1)$ which represents the weight on $f_{1/2}$ in the convex combination)

$$f = \frac{2-t}{2}\chi_{[0,1/2]} + \frac{t}{2}\chi_{(1/2,1]}.$$

This function is left-heavy (i.e. satisfies the conclusion of the Lemma with $n = 1$).

Now assume that every $f \in D_{n-1}$ is such that for every integer k between 0 and 2^{n-2}

$$f(x) \geq f\left(x + \frac{1}{2^{n-1}}\right) \text{ for all } x \in \left(\frac{2k}{2^{n-1}}, \frac{2k+1}{2^{n-1}}\right).$$

Take any appropriate integer k and any x between $\frac{2k}{2^n}$ and $\frac{2k+1}{2^n}$. Take any $g \in D_n$. Note that g is a convex combination of h and $T(f)$ where $h, f \in D_{n-1}$. Since h satisfies the stronger condition

$$h(x) \geq h\left(x + \frac{1}{2^{n-1}}\right),$$

we just need to check $T(f)$.

There are two cases. First, when $x < 1/2$

$$Tf(x) = 2f(2x) \wedge 1$$

and

$$Tf\left(x + \frac{1}{2^n}\right) = 2f\left(2x + \frac{1}{2^{n-1}}\right) \wedge 1.$$

The former of these is no smaller than the latter because our inductive hypothesis applies to f and $2x$.

The second case concerns $x > 1/2$. In this case

$$Tf(x) = (2f(2x-1) - 1) \vee 0$$

and

$$Tf\left(x + \frac{1}{2^n}\right) = \left(2f\left(2x + \frac{1}{2^{n-1}} - 1\right) - 1\right) \vee 0.$$

Again, we have the desired condition because f and $2x - 1$ satisfy the inductive hypothesis.

□

Lemma I.35. *Every $f \in D_n$ is such that for every $m \in \mathbb{N}$ with $m < n$ and every integer k between 0 and $2^{m-1} - 1$*

$$f(x) \geq f\left(x + \frac{1}{2^m}\right) \text{ for all } x \in \left(\frac{2k}{2^m}, \frac{2k+1}{2^m}\right).$$

Proof. When n=1 there is nothing to check. Proceed by induction on n. Fix some $n \in \mathbb{N}$ and assume that the condition holds for D_n.

Let $f \in D_{n+1}$ be given. Consider $m < n+1$. The function f is a convex combination of g and Th where g and h are in D_n. We know by Lemma I.34 and the inductive assumption that the desired condition holds for g. So we just need to check Th.

First consider $k \leq 2^{m-2} - 1$ and $x \in \left(\frac{2k}{2^m}, \frac{2k+1}{2^m}\right)$. Note that this corresponds to $x < 1/2$ and we also have $x + 1/2^m < 1/2$. Compare

$$Th(x) = 2h(2x) \wedge 1$$

and

$$Th\left(x + \frac{1}{2^m}\right) = 2h\left(2x + \frac{1}{2^{m-1}}\right) \wedge 1.$$

Because $h \in D_n$ and $m - 1 < n$ we have $Th(x) \geq Th\left(x + \frac{1}{2^m}\right)$.

Similarly, whenever $k \geq 2^{m-2}$ and $x \in \left(\frac{2k}{2^m}, \frac{2k+1}{2^m}\right)$ we have

$$Th(x) = (2h(2x-1) - 1) \vee 0$$

and

$$Th\left(x + \frac{1}{2^m}\right) = \left(2h\left(2x - 1 + \frac{1}{2^{m-1}}\right) - 1\right) \vee 0.$$

Again, the former is no smaller than the latter because $h \in D_n$ and $n > m - 1$ so we can apply our inductive hypothesis to the variable $2x - 1$. □

Theorem I.36. *Let $f \in D_\infty$. Let $n \in \mathbb{N}$ be given. Then*

$$f(x) \geq f\left(x + \frac{1}{2^n}\right) \text{ for all } x \in \left(\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

whenever $k \in \mathbb{Z}$ is between 0 and $2^{n-1} - 1$.

Proof. It follows from Lemmas I.34 and I.35 and the fact that D_∞ functions are approximated by D_j functions in 1-norm. □

C. THE MAP $T\Delta$

In this section we answer question I.26 and prove that the map $P : C_{1/2} \to C_{1/2}$ given by

$$P(f) = \sum_{n=0}^{\infty} \frac{(T\Delta)^n(f)}{2^{n+1}}$$

is contractive and fixed point free. Note that $T\Delta$ can be written as ([19])

$$T\Delta f(t) = \begin{cases} 2f(4t) \wedge 1 & : 0 \le t < 1/4, \\ 2(1 - f(4t-1)) \wedge 1 & : 1/4 \le t < 1/2, \\ (2f(4t-2) - 1) \vee 0 & : 1/2 \le t < 3/4, \\ (1 - 2f(4t-3)) \vee 0 & : 3/4 \le t \le 1. \end{cases}$$

We start by making new conventions for the notation used in the proof of Theorem I.20. For any $f \in C_{1/2}$ (again, we brush certain measure theoretic concerns under the rug), define

$$A_n(f) = \{x \in [0,1] : (T\Delta)^n f(x) \in (0,1)\}.$$

We begin the proof by adopting Lemma I.21 to our current situation.

Lemma I.37. *For every $f \in C_{1/2}$, $m(A_n(f)) \le \frac{1}{2^n}$.*

Proof. Take any $f \in C_{1/2}$. As with Lemma I.21, we begin by decomposing the set (using A_1 instead of $A_1(f)$ where our meaning is clear)

$$A_1 = (A_1 \cap (0,1/4)) \cup (A_1 \cap (1/4,1/2)) \cup (A_1 \cap (1/2,3/4)) \cup (A_1 \cap (3/4,1)).$$

Zooming in on each of the four pieces and examining the definition of $T\Delta$ gives

$$x \in (A_1 \cap (0,1/4)) \quad \Leftrightarrow \quad f(4x) \in (0,1/2) \text{ and } x \in (0,1/4);$$

$$x \in (A_1 \cap (1/4,1/2)) \quad \Leftrightarrow \quad f(4x-1) \in (1/2,1) \text{ and } x \in (1/4,1/2);$$

$$x \in (A_1 \cap (1/2,3/4)) \quad \Leftrightarrow \quad f(4x-2) \in (1/2,1) \text{ and } x \in (1/2,3/4);$$

$$x \in (A_1 \cap (3/4,1)) \quad \Leftrightarrow \quad f(4x-3) \in (0,1/2) \text{ and } x \in (3/4,1).$$

From these we get the four equations $m(A_1 \cap (0,1/4)) = \frac{1}{4}m[f \in (0,1/2)]$, $m(A_1 \cap (1/4,1/2)) = \frac{1}{4}m[f \in (1/2,1)]$, $m(A_1 \cap (1/2,3/4)) = \frac{1}{4}m[f \in (1/2,1)]$, and $m(A_1 \cap (3/4,1)) = \frac{1}{4}m[f \in (0,1/2)]$. From this it follows that

$$\begin{aligned} m(A_0(f)) &= m[f \in (0,1/2)] + m[f = 1/2] + m[f \in (1/2,1)] \\ &\geq m[f \in (0,1/2)] + m[f \in (1/2,1)] \\ &= 2m(A_1 \cap (0,1/4)) + 2m(A_1 \cap (1/4,1/2)) \\ &+ 2m(A_1 \cap (1/2,3/4)) + 2m(A_1 \cap (3/4,1)) \\ &= 2m(A_1(f)). \end{aligned}$$

Since $f \in C_{1/2}$ was arbitrary, this same inequality applies for any n to $(T\Delta)^{n-1}f$, giving

$$m(A_n(f)) = m(A_1((T\Delta)^{n-1}f)) \leq \frac{1}{2}m(A_0((T\Delta)^{n-1}f)) = \frac{1}{2}m(A_{n-1}(f)).$$

This implies that $m(A_n(f)) \leq \frac{1}{2^n}m(A_0(f)) \leq \frac{1}{2^n} \to 0$. $\square$

Lemma I.38. *Let $h \in C_{1/2}$ be given. If $h(y) = 0$, then*

$$T\Delta h\left(\frac{y}{4}\right) = 0,\ T\Delta h\left(\frac{1}{4} + \frac{y}{4}\right) = 1,\ T\Delta h\left(\frac{1}{2} + \frac{y}{4}\right) = 0,\ \text{and } T\Delta h\left(\frac{3}{4} + \frac{y}{4}\right) = 1.$$

If $h(y) = 1$, then

$$T\Delta h\left(\frac{y}{4}\right) = 1,\ T\Delta h\left(\frac{1}{4} + \frac{y}{4}\right) = 0,\ T\Delta h\left(\frac{1}{2} + \frac{y}{4}\right) = 1,\ \text{and } T\Delta h\left(\frac{3}{4} + \frac{y}{4}\right) = 0.$$

Note that this is not as strong as Lemma I.22. The nature of Alspach's map is that it copies charecteristic functions in a highly regular way. By contrast, $T\Delta$ partially reverses characteristic functions. This reversal is why $T\Delta$ maps C into $C_{1/2}$, but it also makes a direct application of some of the techniques of Theorem I.20 impossible.

Lemma I.39. *P is contractive.*

Proof. Let $f, g \in C_{1/2}$ be given with $\|f-g\|_1 > 0$. As with R from Theorem I.20, it suffices to show that there is some N so that $\left\|\frac{(I+(T\Delta)^N)}{2}f - \frac{(I+(T\Delta)^N)}{2}g\right\|_1 < \|f-g\|_1$. As before, we define

$$B_n = A_n(f) \cup A_n(g),$$

$$C_n = \{x \in [0,1] : ((T\Delta)^n f)(x) = 1 \text{ and } ((T\Delta)^n g)(x) = 0\},$$

$$D_n = \{x \in [0,1] : ((T\Delta)^n f)(x) = ((T\Delta)^n g)(x) \in \{0,1\}\},$$

$$E_n = \{x \in [0,1] : ((T\Delta)^n f)(x) = 0 \text{ and } ((T\Delta)^n g)(x) = 1\}.$$

As before, we just need some N so that $m([f > g] \cap E_N) > 0$ or $m([f < g] \cap C_N) > 0$. Lemma I.37 gives that $m(B_n) \to 0$. From an argument identical to that found in the proof of Lemma I.23 we get that $m(C_n) \to \frac{1}{2}\|f-g\|_1$ and $m(E_n) \to \frac{1}{2}\|f-g\|_1$.

Again, the recursive relation for $T\Delta$ is not as nice as that for T, but we can still apply Lemma I.38 to our sets E_n and C_n to get

$$C_{n+1} \supseteq \frac{1}{4}C_n \cup \left(\frac{1}{4} + \frac{1}{4}E_n\right) \cup \left(\frac{1}{2} + \frac{1}{4}C_n\right) \cup \left(\frac{3}{4} + \frac{1}{4}E_n\right),$$

$$E_{n+1} \supseteq \frac{1}{4}E_n \cup \left(\frac{1}{4} + \frac{1}{4}C_n\right) \cup \left(\frac{1}{2} + \frac{1}{4}E_n\right) \cup \left(\frac{3}{4} + \frac{1}{4}C_n\right).$$

We could continue inductively to recognize that E_{n+k} and C_{n+k} contain $4^{k-1} * 2$ shrunken copies each of E_n and C_n. We will risk confusion and use the notation $E_{n,k,j}$ to stand for either E_n or C_n where

$$E_{n+k} \supseteq \bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k}E_{n,j,k}\right).$$

Similarly, we can write

$$C_{n+k} \supseteq \bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k}C_{n,j,k}\right).$$

Now we will find N. Start by letting n be sufficiently large that $m(E_n)$ and $m(C_n)$ are both greater than $\frac{1}{4}\|f-g\|_1$.

(♣) Claim. There exists $k \in \mathbb{N}$ such that

$$S_1 := E_{n+k} \cap [f > g]$$

has positive measure.

[Proof of (♣)] Let $W := [f > g]$. Fix $\varepsilon > 0$. Again by Royden [46], Chapter 3, Proposition 15, there exists a finite sequence of open intervals $(I_l)_{l=1}^{\nu}$ such that for $\Gamma := \cup_{l=1}^{\nu} I_l$, $m(W \,\Delta\, \Gamma) < \varepsilon$. (Here Δ is the symmetric difference of sets and is not the map from this section.) Without loss of generality, we may assume $(I_l)_{l=1}^{\nu}$ is pairwise disjoint, and that each interval I_l is a dyadic interval of the form $(j_l/4^k, (j_l+1)/4^k)$, for some $j_l \in \{0, \ldots, 4^k - 1\}$, and some $k \in \mathbb{N}$. We may write

$$\chi_\Gamma = \sum_{j=0}^{4^k-1} \beta_j \, \chi_{(j/4^k,(j+1)/4^k)} \ ,$$

where each $\beta_j \in \{0, 1\}$. Then

$$\begin{aligned}
&m(E_{n+k} \cap W) \geq m\left(\bigcup_{j=0}^{4^k-1}\left(\frac{j}{4^k} + \frac{1}{4^k}E_{n,j,k}\right) \cap W \cap \Gamma\right) \\
\geq\ & m\left(\bigcup_{j=0}^{4^k-1}\left(\frac{j}{4^k} + \frac{1}{4^k}E_{n,j,k}\right) \cap \Gamma\right) - m\left(\bigcup_{j=0}^{4^k-1}\left(\frac{j}{4^k} + \frac{1}{4^k}E_{n,j,k}\right) \cap \Gamma \backslash W\right) \\
\geq\ & \int_0^1 \sum_{j=0}^{4^k-1} \chi_{\left(\frac{j}{4^k}+\frac{1}{4^k}E_{n,j,k}\right)} \sum_{s=0}^{4^k-1} \beta_s \, \chi_{\left(\frac{s}{4^k},\frac{s+1}{4^k}\right)} \, dm - m\left(\Gamma \backslash W\right) \\
>\ & \int_0^1 \sum_{j=0}^{4^k-1} \beta_j \, \chi_{\left(\frac{j}{4^k}+\frac{1}{4^k}E_{n,j,k}\right)} \, dm - \varepsilon = \frac{1}{4^k} \sum_{j=0}^{4^k-1} \beta_j m(E_{n,j,k}) - \varepsilon \\
\geq\ & \frac{1}{4}\|f-g\|_1 \frac{1}{4^k} \sum_{j=0}^{4^k-1} \beta_j - \varepsilon = \frac{1}{4}\|f-g\|_1 \, m(\Gamma) - \varepsilon \\
>\ & \frac{1}{4}\|f-g\|_1 \left(m(W) - \varepsilon\right) - \varepsilon \geq \frac{1}{4}\|f-g\|_1 \, m(W) - 2\,\varepsilon \\
>\ & \frac{\|f-g\|_1}{8} \, m(W) > 0 \ ,
\end{aligned}$$

for $\varepsilon \in (0, \infty)$ chosen small enough.

[End proof of (♣)]

Note that ♣ can be modified to give that $S_2 := C_{n+k} \cap [f < g]$ also has positive measure. We will not need this fact, so it is omitted. As before, letting $N = n + k$, we can compute the cancellation. Let $S_3 = [0,1]\backslash(S_1 \cup S_2)$.

$$\begin{aligned}
\left\| \frac{I+(T\Delta)^N}{2} f - \frac{I+(T\Delta)^N}{2} g \right\|_1 &= \int_0^1 \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \left| \frac{f-g-1}{2} \right| + \int_{S_2} \left| \frac{f+1-g}{2} \right| &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \frac{1+g-f}{2} + \int_{S_2} \frac{1+f-g}{2} &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
< \int_{S_1} \frac{1+f-g}{2} + \int_{S_2} \frac{1+g-f}{2} &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \left(\left| \frac{(T\Delta)^N f - (T\Delta)^N g}{2} \right| + \left| \frac{f-g}{2} \right| \right) &+ \int_{S_2} \left(\left| \frac{(T\Delta)^N f - (T\Delta)^N g}{2} \right| + \left| \frac{f-g}{2} \right| \right) \\
&+ \int_{S_3} \left| \frac{f-g}{2} + \frac{(T\Delta)^N f - (T\Delta)^N g}{2} \right| \\
\leq \int_0^1 \left| \frac{f-g}{2} \right| &+ \left| \frac{(T\Delta)^N f - (T\Delta)^N g}{2} \right| = \|f-g\|_1.
\end{aligned}$$

□

We can re-arrange P the same way that we re-arranged R.

$$\begin{aligned}
Pf &= \frac{f}{2} + \frac{T\Delta f}{4} + \frac{(T\Delta)^2 f}{8} + \frac{(T\Delta)^3 f}{16} + \cdots \\
&= \frac{f}{2} + \frac{T\Delta f}{4} + \frac{(T\Delta)(T\Delta f)}{8} + \frac{(T\Delta)^2 (T\Delta f)}{16} + \cdots \\
&= \frac{f}{2} + \frac{1}{2} \left(\frac{I}{2}(T\Delta f) + \frac{T\Delta}{4}(T\Delta f) + \frac{(T\Delta)^2}{8}(T\Delta f) + \cdots \right) \\
&= \frac{I}{2} f + \frac{P}{2} T\Delta f.
\end{aligned}$$

Lemma I.40. *P is fixed point free.*

Proof. This proof will be nearly identical to that of Lemma I.25. Note that because $P : C \to C_{1/2}$, we are actually showing that P is fixed point free on C.

Start by taking $f, g \in C_{1/2}$ with $\|f-g\|_1 > 0$. The contractivity of P gives that

$$\|PT\Delta f - PT\Delta g\|_1 < \|T\Delta f - T\Delta g\|_1 = \|f-g\|_1.$$

But then using the self-similar rearrangement of P we get

$$\begin{aligned}\|Pf - Pg\|_1 &= \left\|\frac{f-g}{2} + \frac{PT\Delta f - PT\Delta g}{2}\right\|_1 \\ &\geq \left\|\frac{f-g}{2}\right\|_1 - \left\|\frac{PT\Delta f - PT\Delta g}{2}\right\|_1 \\ &> 0.\end{aligned}$$

Assuming that f_0 is any fixed point of P we get

$$f_0 = Pf_0 = \frac{f_0}{2} + \frac{T\Delta f_0}{4} + \frac{(T\Delta)^2 f_0}{8} + \frac{(T\Delta)^3 f_0}{16} + \cdots$$

$$\Rightarrow \frac{f_0}{2} = \frac{T\Delta f_0}{4} + \frac{(T\Delta)^2 f_0}{8} + \frac{(T\Delta)^3 f_0}{16} + \cdots$$

$$\Rightarrow f_0 = \frac{T\Delta f_0}{2} + \frac{(T\Delta)^2 f_0}{4} + \frac{(T\Delta)^3 f_0}{8} + \cdots = P(T\Delta f_0).$$

Then, P being 1-1, we have $T\Delta f_0 = f_0$. This contradicts Lemma 1 of [19]. □

Theorem I.41. *P is contractive and fixed point free on $C_{1/2}$.*

This theorem is the direct consequence of Lemmas I.39 and I.40.

Open Question I.42. *In light of Theorem I.41, can we use the techniques of [19] to show that every subset of $L^1[0,1]$ that contains the nontrivial intersection of an order interval and finitely many hyperplanes admits a fixed point free contractive map?*

D. GEOMETRIC SERIES NOT INVOLVING ALSPACH'S MAP

We want to know which properties of T and $T\Delta$ lead to the result that $\sum_{n=0}^{\infty} \frac{T^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(T\Delta)^n}{2^{n+1}}$ are contractive and fixed point free.

It is not enough to have $S : C_{1/2} \to C_{1/2}$ be nonexpansive such that $\frac{I+S}{2}$ contracts some, but not all, pairs of $C_{1/2}$ elements.

Consider the map $S : C_{1/2} \to C_{1/2}$ given by

$$(Sf)(x) = \begin{cases} f(x) & , 0 \le x < \frac{1}{2} \\ 2f(2x - 1/2) \wedge 1 & , \frac{1}{2} \le x < \frac{3}{4} \\ (2f(2x-1) - 1) \vee 0 & , \frac{3}{4} \le x < 1. \end{cases}$$

S has two distinct fixed points, $\chi_{(0,1/2)}$ and $\chi_{(1/2,1)}$. So $\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} S^k$ has the same fixed points and fails to contract the pair.

And yet $\frac{I+S}{2}$ contracts some pairs of points. For example the pair $f_1 = \frac{1}{2}\chi_{(0,1/2)} + \chi_{(1/2,3/4)}$, $f_2 = \frac{1}{2}\chi_{(0,1/2)} + \chi_{(3/4,1)}$ is such that $\|f_1 - f_2\|_1 = 1/2$ and yet $\|\frac{I+S}{2} f_1 - \frac{I+S}{2} f_2\|_1 = 1/4$.

Open Question I.43. *For which nonexpansive $F : C_{1/2} \to C_{1/2}$, is $\displaystyle\sum_{n=0}^{\infty} \frac{F^n}{2^{n+1}}$ contractive and fixed point free?*

Conjecture I.44. *Suppose $S : C_{1/2} \to C_{1/2}$ is such that there exists an $f_0 \in C_{1/2}$ with $S(f_0) \neq f_0$ such that for every $f \in C_{1/2}$ we have $S^n(f) \xrightarrow[n]{weak} f_0$. Then $\displaystyle\sum_{n=0}^{\infty} \frac{S^n}{2^{n+1}}$ is contractive and fixed point free.*

E. ITERATES LEAD TO FIXED POINTS

The previously mentioned Banach Contraction Mapping Theorem is proved by examining the iterates of the given contraction mapping. This idea is used in many different ways throughout metric fixed point theory. For example, Corollary 9.1 on page 101 of [27] says

Theorem I.45. *Let X be a Banach space, let $K \subseteq X$ be closed and convex, and let $T : K \to K$ be a nonexpansive mapping for which $T(K)$ is compact. Then for each $\alpha \in (0,1)$ the mapping $T_\alpha = \alpha I + (1-\alpha)T$ is such that the iterates of T_α converge to a fixed point of T.*

Sometimes it does not suffice to look only at a sequence of finite iterates. Chapter 14 of [27] has a very nice discussion of transfinite iterates and applications to fixed point theory. In particular, we were able to generalize Theorem 14.10 of [27]. First we need some details about the mechanism. Note that the authors of [27] use nets in their transfinite induction. We will use filters instead.

Definition I.46. *Let S be a non-empty set. A **filter** $\mathcal{F}$ in S is a collection of non-empty subsets of S satisfying*

- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}$ *and* $A \subseteq B \Rightarrow B \in \mathcal{F}$

Definition I.47. *An **ultrafilter** is a filter that is maximal with respect to inclusion. That is, $\mathcal{F}$ is an ultrafilter in S if for every filter $\mathcal{G}$ in S with $\mathcal{G} \supseteq \mathcal{F}$ we have $\mathcal{F} = \mathcal{G}$.*

We can use ultrafilters to generalize the concept of a sequential limit. In some sense ultrafilters are "longer convergence objects" than sequences. They appear throughout mathematics. For example, there is a very elegant and accessible approach to nonstandard analysis that uses ultrafilters (see section 1.2 of [38]).

Definition I.48. *Let X be a topological space and let $\mathcal{U}$ be an ultrafilter on an index set I. Let $x : I \to X$ and for each i in I denote $x(i) = x_i$. Then $\{x_i : i \in I\}$ is said to **converge with respect to $\mathcal{U}$** to an element $y \in X$ if for each neighborhood V of y the set $\{i \in I : x_i \in V\}$ is an element of $\mathcal{U}$. When this is the case we call y an **ultralimit** and write*

$$\lim_{\mathcal{U}} x_i = y \qquad \textit{or} \qquad \mathcal{U}\lim x_i = y.$$

Recall the famous Bolzano-Weierstrass Theorem that says all sequences in closed and bounded subsets of Euclidean spaces have convergent subsequences. We teach our advanced calculus students the generalization that says that all sequences in compact metric spaces have convergent subsequences.

One of the properties of ultrafilters is that they are selective. If you apply an ultralimit in one of these settings, the ultralimit will choose one of the subsequential limits. The following result (Theorem 14.1 in [27]) is in the same vein. Note that the following result is not restricted to metric topologies. This will become important when we consider the weak topology.

Theorem I.49. *Let X be a compact Hausdorff space. Let $\mathcal{U}$ be an ultrafilter in the index set I. Suppose $\{x_i\}_{i\in I} \subseteq X$. Then there exists a unique $y \in X$ such that*

$$\mathcal{U} \lim x_i = y.$$

Now we can define the transfinite iterates. We will adhere to the notation of [27] for ordinal arithmetic. Let $\Omega = \Omega_0 \cup \{\omega_1\}$, where Ω_0 is the set of all countable ordinals and ω_1 is the first uncountable ordinal. Note that $(\Omega, \leq)$ is totally ordered.

Now consider any weakly compact and convex K and any nonexpansive $T : K \to K$. For every $n \in \mathbb{N} \subseteq \Omega$ we define T^n in the natural way: $T^n(x) = T(T^{n-1}(x))$ for every $x \in K$, which is defined using usual induction. This suggests our way forward.

Let $\alpha \in \Omega$ be given so that $\forall \gamma \in \Omega$ with $\gamma < \alpha$, $T^\gamma(x)$ is defined for all $x \in K$. If $\alpha = \beta + 1$ for some $\beta \in \Omega$, then define $T^\alpha(x) = T(T^\beta(x))\ \forall x \in K$. Such α are called **successor ordinals**.

Otherwise, α is called a **limit ordinal**. In this case we let $I = \{\beta \in \Omega : \beta < \alpha\}$. Let $\mathcal{U}$ be an ultrafilter in I. Because K is weakly compact it follows from Theorem I.49 that we can define for any $x \in K$

$$T^\alpha(x) = \mathcal{U} \lim_{\beta \in I} T^\beta(x).$$

Note that this limit is a weak limit, i.e. a limit with respect to the weak topology. It follows from the above and the Principle of Transfinite Induction that the iterates T^α are well defined for all $\alpha \in \Omega$. Also, we note that the index set I depends on α. For each limit ordinal α, the fact that the corresponding index set I admits an ultrafilter follows from Zorn's Lemma (the proof is identical to the proof that $\mathbb{N}$ admits an ultrafilter).

Lemma I.50. *If T is nonexpansive, then for every $\alpha > \gamma \in \Omega$ it follows that $\|T^\alpha(x) - T^\alpha(z)\| \leq \|T^\gamma(x) - T^\gamma(z)\|$. Note that this gives that T^α is nonexpansive for every α.*

Proof. Let x and z be fixed. We are given that $T = T^1$ is nonexpansive. This will serve as our base case.

Next, let $\beta \in \Omega$ be given and suppose by way of induction that for every $\alpha > \gamma \in \Omega$ with $\beta > \alpha$ we have that $\|T^\alpha(x) - T^\alpha(z)\| \leq \|T^\gamma(x) - T^\gamma(z)\|$.

Here we have two cases. If β is a successor ordinal, then letting $\alpha = \beta - 1$ in the inductive hypothesis gives that for every $\gamma < \beta$ we have (from the inductive hypothesis and the fact that T is nonexpansive) that

$$\|T^{\beta}(x) - T^{\beta}(z)\| = \|T(T^{\alpha})(x) - T(T^{\alpha})(z)\| \leq \|T^{\alpha}(x) - T^{\alpha}(z)\| \leq \|T^{\gamma}(x) - T^{\gamma}(z)\|.$$

Now suppose that β is a limit ordinal. By way of contradiction, suppose that there is some $\gamma < \beta$ such that $\|T^{\beta}(x) - T^{\beta}(z)\| > \|T^{\gamma}(x) - T^{\gamma}(z)\|$. Then for every α with $\gamma < \alpha < \beta$ we would have

$$\|T^{\beta}(x) - T^{\beta}(z)\| > \|T^{\gamma}(x) - T^{\gamma}(z)\| \geq \|T^{\alpha}(x) - T^{\alpha}(z)\|.$$

Now we can generate a contradiction. First, recall the definition

$$\|T^{\beta}(x) - T^{\beta}(z)\| = \|\mathcal{U}\lim_{\alpha<\beta} T^{\alpha}(x) - \mathcal{U}\lim_{\alpha<\beta} T^{\alpha}(z)\|.$$

Because ultralimits respect the same algebraic laws as regular limits we can re-write this as

$$\|T^{\beta}(x) - T^{\beta}(z)\| = \|\mathcal{U}\lim_{\alpha<\beta}(T^{\alpha}(x) - T^{\alpha}(z))\|.$$

Now, recalling that these ultralimits are with respect to the weak topology, the weak lower semicontinuity of the norm gives that

$$\|T^{\beta}(x) - T^{\beta}(z)\| = \|\mathcal{U}\lim_{\alpha<\beta}(T^{\alpha}(x) - T^{\alpha}(z))\| \leq \mathcal{U}\lim_{\alpha<\beta} \|T^{\alpha}(x) - T^{\alpha}(z)\|.$$

Finally, our contradiction comes from the inductive hypothesis:

$$\begin{aligned} \|T^{\beta}(x) - T^{\beta}(z)\| &\leq \mathcal{U}\lim_{\alpha<\beta} \|T^{\alpha}(x) - T^{\alpha}(z)\| \\ &\leq \mathcal{U}\lim_{\alpha<\beta} \|T^{\gamma}(x) - T^{\gamma}(z)\| \\ &= \|T^{\gamma}(x) - T^{\gamma}(z)\| < \|T^{\beta}(x) - T^{\beta}(z)\|. \end{aligned}$$

$\square$

Theorem I.51. *[Theorem 14.10 of [27]] Suppose X is a Banach space which is either strictly convex or has Kadec-Klee norm. Suppose K is a weakly compact convex subset of X. Suppose $T : K \to K$ is nonexpansive with at least one fixed point. Let $F = (I+T)/2$. If $\{F^\alpha\}$ is the collection of transfinite iterates of F, then for every $x \in K$ there is some $\alpha \in \Omega$ such that $F^\alpha x$ is a fixed point of T.*

We can prove an analogous result for contractive maps on all weakly compact convex sets. Note that in light of Theorem I.20, the assumption that $Fix(T) \neq \phi$ is necessary.

Theorem I.52. *Let $(X, \|\cdot\|)$ be a Banach space. Let $C \subseteq X$ be non-empty, weakly compact, and convex. Let $T : C \to C$ be contractive with a fixed point z. Then for every $x_0 \in C$ there is some countable ordinal $\alpha \in \Omega$ such that $T^\alpha(x_0) = z$.*

Proof. Fix $x \in C$. First, I.50 tells us that for every $\alpha > \gamma \in \Omega$ it follows that $\|T^\alpha(x) - T^\alpha(z)\| \leq \|T^\gamma(x) - T^\gamma(z)\|$.

Suppose, by way of contradiction, that $T^\alpha x \neq z$ for every $\alpha \in \Omega$.

Because T is contractive and z is a fixed point, we have that $\|T^\beta(x) - (z)\| = \|T^\beta(x) - T^\beta(z)\| < \|T^{\beta-1}(x) - T^{\beta-1}(z)\| = \|T^{\beta-1}(x) - (z)\|$ for every successor ordinal β (Again, we are supposing that the right hand side of this inequality is positive.)

Let $r = \|x - z\|$. We know that there are uncountably many successor ordinals in Ω (if there were uncountably many limit ordinals $\{\alpha\}$ only, then the collection $\{\alpha + 1\}$ would be uncountable). Define B_n (the B stands for "bin") to be the set

$$B_n = \{\beta \in \Omega : \|T^{\beta-1}x - T^{\beta-1}z\| - \|T^\beta x - T^\beta z\| > 1/n\}.$$

Notice that at least one of these bins has to be infinite (actually uncountable, but we just need infinite) because $\cup B_n$ is the uncountable collection of successor ordinals. Fix N so that B_N is infinite. Choose an integer M so that $M > N \cdot r$. Choose any M elements from B_N.

Label them in order so that $\beta_1 < \beta_2 < \cdots < \beta_M$. Take any $\alpha > \beta_M$.

$$\begin{aligned}
\|T^\alpha x - z\| &\leq \|T^{\beta_M} x - z\| \leq \|T^{\beta_M - 1} x - z\| - \frac{1}{N} \\
&\leq \|T^{\beta_{M-1}} x - z\| - \frac{1}{N} \leq \|T^{\beta_{M-1}-1} x - z\| - \frac{1}{N} - \frac{1}{N} \\
&\leq \|T^{\beta_{M-2}} x - z\| - \frac{2}{N} \leq \|T^{\beta_{M-2}-1} x - z\| - \frac{1}{N} - \frac{2}{N} \\
&\leq \cdots \\
&\leq \|T^{\beta_1} x - z\| - \frac{M-1}{n} \leq \|T^{\beta_1 - 1} x - z\| - \frac{1}{N} - \frac{M-1}{n} \\
&\leq \|x - z\| - \frac{M}{N} = r - \frac{M}{N}.
\end{aligned}$$

Because this last quantity is negative, we have our contradiction. □

F. FUTURE WORK AND OPEN QUESTIONS

1. Renorming c_0

Example 1: The non-reflexive Banach space $\ell^1(\mathbb{R})$ fails the fpp(n.e.). Let $\{e_n\}$ be the usual Schauder basis for ℓ^1. Define $C = \overline{co}\{e_n\}$. It turns out that this implies

$$C = \{x = (t_n) = \sum_{\mathbb{N}} t_n e_n : \text{ each } t_n \geq 0, \text{ and } \sum_{\mathbb{N}} t_n = 1\}.$$

C is a closed, bounded, and convex set.

Now define $T : C \to C$ to be a right shift map given by

$$T(t_1, t_2, ..., t_n, ...) = (0, t_1, ..., t_{n-1}, ...).$$

We can also write

$$T\left(\sum_{j=1}^{\infty} t_j e_j\right) = \sum_{j=1}^{\infty} t_j e_{j+1}.$$

It follows that $\forall t, u \in C$

$$\|T(t) - T(u)\|_1 = \sum_{j=1}^{\infty} |t_j - u_j| = \|u - t\|_1.$$

T is an isometry (and therefore nonexpansive).

To see that T is fixed point free, suppose $t \in C$ and $T(t) = t$. Then $t_1 = 0$, $t_2 = t_1 = 0$, and $t_n = 0$ for all $n \in \mathbb{N}$. This contradicts the fact that $\sum t_n = 1$.

Example 2: $(c_0(\mathbb{R}), \|\cdot\|_\infty)$ is another non-reflexive Banach space failing fpp(n.e.).

Define $C = \{(s_n \in c_0) : 1 = s_1, \text{ and } s_n \geq s_{n+1} \, \forall n\}$. This set is closed, bounded, and convex. Define $S : C \to C$ to be a right shift map given by

$$S(s_1, s_2, s_3, ..., s_n, ...) = (1, s_1, s_2, ..., s_{n-1}, ...).$$

It is straightforward to check that $\|S(u) - S(w)\|_\infty = \|u - w\|_\infty \; \forall u, w \in c_0$, i.e. S is an isometry.

To see that $S : C \to C$ is fixed point free, suppose that $S(s) = s$ for some $s \in C \subset c_0$. Then $s_1 = 1$, $s_2 = s_1 = 1$, and $s_n = s_{n-1} = 1$ for all $n \in \mathbb{N}$. But this means $s \notin c_0$.

We can also write down this last example in a way that is more closely analogous to example 1.

Example 3: Let $\eta_1 = e_1$, $\eta_2 = e_1 + e_2$, and $\eta_n = \sum_{k=1}^n e_k$. This sequence (η_n) is the summing basis, or Schauder basis, for c_0.

Fact: C from Example 2 can be written as both

$$A = \overline{co}\{\eta_n\} \quad \textbf{and} \quad B = \{t = \sum_{\mathbb{N}} \alpha_n \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{\mathbb{N}} \alpha_n = 1\}.$$

Also, the map $S : C \to C$ in the previous example can be re-written as

$$S\left(\sum_{n=1}^{\infty} \alpha_n \eta_n\right) = \sum_{n=1}^{\infty} \alpha_n \eta_{n+1}.$$

Claim: $C \subseteq B$

Proof. Let $(s_n) \in C$ be given. Define $\alpha_n = s_n - s_{n+1}$ $\forall n \in \mathbb{N}$. Note that $s_N \to 0$ implies that

$$\sum_{k=1}^{N} \alpha_k = s_1 - s_{N+1} \underset{N}{\longrightarrow} s_1 = 1.$$

Then

$$\sum_{k=1}^{\infty} s_k e_k = \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \alpha_n \right) e_k = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \alpha_n e_k =$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \alpha_n e_k = \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{n} e_k = \sum_{n=1}^{\infty} \alpha_n \eta_n \in B.$$

□

Claim: $B \subseteq A$.

Proof. Let $x = \sum_1^\infty \alpha_n \eta_n$ be given. Then the sequence $x_N = \sum_{n=1}^{N} \alpha_n \eta_n$ converges to x. But we do not necessarily have $x_N \in co\{\eta_n\}$. Note that $\sum_{N+1}^{\infty} \alpha_n$ converges to 0 as $N \to \infty$. Now define

$$\tilde{x}_N = x_N + \left(\sum_{N+1}^{\infty} \alpha_N \right) \eta_1.$$

Now $\tilde{x}_N \in co\{\eta_n\}$ and $\tilde{x}_N \to x$. □

Claim: $A \subseteq C$.

This one follows from noticing that $co\{\eta_n\} \subseteq C$ and C is closed.

Claim: $S : C \to C$ in the previous example can be re-written as

$$S \left(\sum_{n=1}^{\infty} \alpha_n \eta_n \right) = \sum_{n=1}^{\infty} \alpha_n \eta_{n+1}$$

Proof. Let $\sum_{n=1}^{\infty} \alpha_n \eta_n$ be given. Using the calculation from the first claim, and defining $s_1 = 1$ and $s_{n+1} = s_n - \alpha_n$, we have

$$S \left(\sum_{n=1}^{\infty} \alpha_n \eta_n \right) = S \left(\sum_{k=1}^{\infty} s_k e_k \right) = e_1 + \sum_{k=1}^{\infty} s_k e_{k+1}.$$

But now this is all equal to

$$e_1 + \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \alpha_n \right) e_{k+1} = e_1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \alpha_n e_{k+1} = e_1 + \sum_{n=1}^{\infty} \alpha_n \sum_{k=2}^{n+1} e_k$$

$$= \sum_{n=1}^{\infty} \alpha_n e_1 + \sum_{n=1}^{\infty} \alpha_n \sum_{k=2}^{n+1} e_k = \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{n+1} e_k = \sum_{n=1}^{\infty} \alpha_n \eta_{n+1}.$$

□

This shows that there is a close analogy between examples 1 and 2.

Example 4: In 2008 (in [37]) Pei-Kee Lin proved that there exists a non-reflexive Banach space $(X, \|\cdot\|)$ with the fixed point property. This Banach space is an equivalent renorming of ℓ^1.

Open Question I.53. *Is there an analogous result for a renorming of c_0?*

Maurey showed ([41]) that if $K \subseteq (c_o, \|\cdot\|_\infty)$ is weakly compact and convex, then K has fpp(n.e.). Dowling, Lennard, and Turett developed a converse ([21]) which says that if K is c.b.c. and not weakly compact, then K fails the fpp-ne.

Borwein and Sims ([5]) proved something similar to Maurey, only for $(c, \|\cdot\|_\infty)$.

Theorem I.54. *If $K \subseteq (c, \|\cdot\|_\infty)$ is weakly compact and convex, then K has the fpp-ne.*

Open Question I.55. *Suppose $K \subseteq (c, \|\cdot\|_\infty)$ is c.b.c. and not weakly compact. Does it follow that K fails fpp-ne?*

There is a partial result in [20].

Theorem I.56. *Suppose $K \subseteq (c, \|\cdot\|_\infty)$ is c.b.c. and not weakly compact. Then there is a $C \subseteq K$ of the same type such that C fails fpp-ne.*

2. Super-Reflexive Spaces

Definition I.57. *A Banach space X is **super-reflexive** if every ultrapower of X is reflexive. Equivalently, X is **super-reflexive** if X has an equivalent uniformly convex norm.*

Maurey proved the existence of fixed points for isometries in super-reflexive spaces.

Theorem I.58. ***[Maurey]*** *Suppose X is a super-reflexive Banach space, and $C \subseteq X$ is closed, bounded, and convex. Let $T : C \to C$ be an isometry. Then T has a fixed point.*

There is a helpful amount of detail in [2]. As mentioned previously, there is also a nice discussion of ultrapowers applied to fixed point theory in [27].

The previous Theorem begs a question, which has been carefully studied and is yet unanswered.

Open Question I.59. *Suppose X is a super-reflexive Banach space, and $C \subseteq X$ is closed, bounded, and convex. Let $T : C \to C$ be nonexpansive. Does it follow that T has a fixed point?*

We will try to generate a result which is stronger that Maurey's but weaker than the question for general nonexpansive maps.

Recall from the main theorem that a map $T : C \to C$ is **contractive** if $\|T(x) - T(y)\| < \|x - y\| \; \forall x, y \in C$.

Definition I.60. *A map $T : C \to C$ is **nowhere contractive** if $\|T(x) - T(y)\| \geq \|x - y\|$ $\forall x, y \in C$.*

Definition I.61. *A map $T : C \to C$ is **asymptotically nonexpansive** if there exists a sequence $\{k_n\}$ with $k_n \to 1$ such that for all $x, y \in C$ we have*

$$\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|.$$

Conjecture I.62. *If X is a super-reflexive Banach space, $C \subseteq X$ c.b.c., and $T : C \to C$ is non-contractive and asymptotically nonexpansive, it follows that T has a fixed point.*

Open Question I.63. *Is the previous claim true?*

Maurey showed ([2]) that if X is a non-reflexive Banach space and $K \subseteq X$ is c.b.c., then K has the fixed point property for isometries.

Open Question I.64. *What about fpp-ne? What about non-contractive? What about non-contractive and asymptotically nonexpansive?*

3. The Isometric and Contractive Parts of a Nonexpansive Map

Sometimes it is easier to guarantee a fixed point for an isometry on certain sets than it is to guarantee a fixed point for a general nonexpansive map on the same sets.

Question I.65. *Is there a way to make sense of the phrase* ***the isometric part*** *of a nonexpansive map?*

Let $(X, \|\cdot\|)$ be a Banach space, with $C \subseteq X$ closed bounded and convex. Let $f : C \to C$ be nonexpansive. Define the isometric part of f to be

$$I(f) = \{(x, y) \in C \times C : \|x - y\| = \|f(x) - f(y)\|\}.$$

If f has a fixed point c, then

$$F : I(f) \to C \times C : (x, y) \to (f(x), f(y))$$

has a fixed point at (c, c). Conversely, F has a fixed point at (d, e) only if f has fixed points at d and e.

We can change the definition of $I(f)$ so that F becomes a self map. Define

$$I^*(f) = \{(x, y) \in C \times C : \|x - y\| = \|f^n(x) - f^n(y)\| \text{ for all } n \in \mathbb{N}\}.$$

But $I(f)$ and $I^*(f)$ contain the entire diagonal of $C \times C$, which is a copy of C. So we haven't shrunk C, we've just made it bigger and weird. Also, F is not an isometry on either domain under the natural product norm.

This doesn't seem to work. Does something else? That is, is there some natural way to take an answer to the fixed point question for isometries and/ or contractive maps on a given set and bootstrap up to the answer to the fixed point question for all nonexpansive maps on that set? At the beginning of this chapter we saw the construction of a fixed point free contractive map that built on a fixed point free isometry. So, it might be justified to see these questions as linked in certain settings.

Question I.66. *Let C be a c.b.c. set in some Banach space X. Suppose that every isometry $F : C \to C$ has a fixed point and every contractive map $G : C \to C$ has a fixed point. Does it follow that every nonexpansive map $T : C \to C$ has a fixed point?*

4. Open Questions From Sections A through E

Question I.29: For which nonexpansive $F : C_{1/2} \to C_{1/2}$, is $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ contractive?

Question I.30: For which nonexpansive $F : C_{1/2} \to C_{1/2}$, is $\displaystyle\sum_{n=0}^{\infty} \frac{(TF)^n}{2^{n+1}}$ fixed point free?

Question I.33: What is the minimal invariant set for R or $\sum \frac{(T\Delta)^n}{2^{n+1}}$?

Open Question I.67. *Given a seed function like $f = \frac{1}{2}\chi_{[0,1]}$, does the sequence $R^n f$ have a weak limit?*

Question I.42: In light of Theorem I.41, can we use the techniques of [19] to show that every subset of $L^1[0,1]$ that contains the nontrivial intersection of an order interval and finitely many hyperplanes admits a fixed point free contractive map?

Question I.43: For which nonexpansive $F : C_{1/2} \to C_{1/2}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{F^n}{2^{n+1}}$ contractive and fixed point free? In particular, is it the case that when $G : C_{1/2} \to C_{1/2}$ is such that there exists an $f_0 \in C_{1/2}$ with $G(f_0) \neq f_0$ such that for every $f \in C_{1/2}$ we have $G^n(f) \xrightarrow[n]{weak} f_0$, does it follow that $\displaystyle\sum_{n=0}^{\infty} \frac{G^n}{2^{n+1}}$ is contractive and fixed point free?

II. SUMMABILITY

A. SUMMABILITY METHODS

Summability theory draws its name from the effort to assign a meaningful value to a sequence of partial sums that may not converge in the traditional sense. The effort dates back at least to Euler's time (see [33, p. vii and 326]). The study has important applications in Number Theory, Analysis, and elsewhere. This is despite the assertion of Niels Abel in 1828 that "divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."

Abel may have been on to something. Before we continue to examine summability theory, consider the following diabolical demonstration.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) \cdots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} + \frac{1}{2n}\right) = \sum_{n=1}^{\infty} \frac{4n-1}{2n(2n-1)}.
\end{aligned}$$

Taking the difference of the two equal quantities gives $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{4n-1}{2n(2n-1)} = 0$. Since all the terms are positive, we can write $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{4n-1}{2n(2n-1)} = 0$. Combining these fractions gives $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = 0$. We can legitimately re-write the left hand side of this last series using partial fractions as $\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n}$. This is now the alternating harmonic series, which is well known to converge to $\ln(2)$. Putting this together gives

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{4n-1}{2n(2n-1)} = 0.$$

Concluding that $\ln(2) = 0$ informs us that we made at least one mistake. Here we erred in treating divergent series as ordinary arithmetic objects. When replacing the difference of series with a series of differences (the mistake in the argument), we noticed that the terms in each sum were positive. If the series converged, this would imply that they converged absolutely, meaning that we had access to some additional arithmetic tools.

This sort of nonsense does not force us to completely avoid divergent sequences and series. When confronted with such confusion and apparent contradiction, we pursue rigor and consistency. In the case of divergent sequences and series, a rigorously founded and consistent tool is called a summability method and is defined as follows.

Definition II.1. *Let $\mathbb{K}$ be a topological space. Let D be some collection of sequences with values in $\mathbb{K}$. A* ***summability method*** *is a function $M : D \to \mathbb{K}$. A given summability method $M : D \to \mathbb{K}$ is called* ***regular*** *if $M((s_N)_{N\in\mathbb{N}}) = \lim_{N\to\infty} s_N$ whenever (s_N) is a convergent sequence in $\mathbb{K}$.*

One of the most important books in summability theory is "Divergent Series" by Hardy [30, 1949]. Like Hardy, we will consider $\mathbb{K}$ to be a vector space such as $\mathbb{R}$ or $\mathbb{C}$. We will also consider only those summability methods, M, that satisfy the conditions:

- if $M((a_n)_{\mathbb{N}}) = s$, then $M((ka_n)_{\mathbb{N}}) = ks$;
- if $M((a_n)_{\mathbb{N}}) = s$ and $M((b_n)_{\mathbb{N}}) = t$, then $M((a_n)_{\mathbb{N}} + (b_n)_{\mathbb{N}}) = s + t$.

One can see that taking ordinary limits of convergent sequences formally satisfies the conditions of a summability method.

The first method that we will consider which assigns values to certain non-convergent sequences will be the Cesàro method. Hardy and others give this method the notation $(C, 1)$.

For any scalar sequence z we define the **Cesàro Average** of z to be the sequence for which the n-th term is

$$(Cz)_n := \frac{1}{n}\sum_{k=1}^{n} z_k \ .$$

If this sequence converges to L, then we say that z is **Cesàro summable** and that the **Cesàro limit** of z is L. We use Ces to denote the space of Cesàro summable bounded sequences. The fact that $(C,1) : Ces \to \mathbb{K}$ is linear follows from the linearity of arithmetic means (here $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

The fact that $(C,1)$ is regular is due to Cauchy [11]. This result is an example of an Abelian theorem. For some, an Abelian theorem is any result showing that a certain method is regular. For us (and [33]), an **Abelian theorem** will be any result showing that some summability method recognizes at least as much convergence as some other method.

To clarify, let M be a summability method with domain D (e.g. $\lim : c \to \mathbb{R}$ where c is the space of convergent sequences). Let N be some other method with domain E (e.g. $(C,1) : Ces \to \mathbb{R}$). The Abelian theorem here would be that for all $(x_n) \in D$ we have $(x_n) \in E$ as well and $N(x_n) = M(x_n)$ (e.g. $(C,1)(x_n) = \lim(x_n)$ for all $(x_n) \in c$). In this case we say that N is a stronger method than M. The fact that gives these results their name is Abel's Theorem, which follows (see [30, p. 149]).

Theorem II.2. *Let $\{a_n\}$ be a sequence of scalar values. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$. If $\sum_{n=0}^{\infty} a_n$ converges to L, then $\lim_{x\to 1^-} F(x) = L$.*

A **Tauberian theorem** is any partial converse to an Abelian theorem. Suppose M, D, N, and E are as in the previous paragraph, with N stronger than M. Let T be some condition satisfied by certain elements of E (called a Tauberian condition). A Tauberian theorem would be as follows: For every $x \in E$ such that $T(x)$ is true, it follows that $x \in D$ and $M(x) = N(x)$.

The Hardy-Littlewood Theorem is possibly the most famous Tauberian theorem. We will come back to this later after seeing some details of our own Tauberian Theorem which is a quantitative version of a corollary to Hardy-Littlewood.

B. BANACH LIMITS

Stefan Banach was also interested in the idea of assigning a kind of limit to sequences and functions that did not have a limit in the usual sense. In [4, ch 2], Banach uses the Hahn-Banach Extension Theorem to construct a generalized limit that we now call a "Banach limit".

Definition II.3. *A **Banach Limit** is a functional $L \in (\ell^\infty)^*$ such that (letting (ξ_n) and (η_n) be arbitrary bounded sequences and letting a and b be some numbers)*

(i) $L(a\xi_n + b\eta_n) = aL(\xi_n) + bL(\eta_n)$

(ii) $L(\xi_n) \geq 0$, *if* $\xi_n \geq 0$ *for all* $n = 1, 2, ...,$

(iii) $L(\xi_2, \xi_3, \xi_4, ...) = L(\xi_1, \xi_2, \xi_3, ...)$,

(iv) $L(1, 1, 1, ...) = 1$.

Banach notes that conditions (i)-(iv) imply that $\liminf x_n \leq L(x_n) \leq \limsup x_n$ for any Banach limit L.

One important contribution to the theory of Banach limits comes from George Lorentz. He noted that there are some sequences which diverge, and yet are taken to the same number by every Banach limit.

Definition II.4. *The bounded sequence (x_n) is called **almost convergent** if there is some number s such that $L(x_n) = s$ for every Banach limit L.*

For example, take the sequence $x = (1, 0, 1, 0, ...)$. Set $y = (0, 1, 0, 1, ...)$. Condition (iii) tells us that $L(x) = L(y)$ for any Banach limit L. But also $L(x) + L(y) = L(x + y) = L(1, 1, 1, ...) = 1$. Therefore $L(x) = 1/2$ for any Banach limit L. So $(1, 0, 1, 0, ...)$ is almost convergent to $1/2$.

Lorentz proved the following theorem in [39, p. 170].

Theorem II.5. *The bounded sequence (x_n) is almost convergent to s if and only if*

$$\lim_{p \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s$$

holds uniformly in n.

It is now easy to see an example of a sequence which is not almost convergent.

Example II.6. *The sequence* $a = (1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, \dots)$*, which has monotonically increasingly long sections of* 1*s and* -1*s in the pattern indicated, fails to be almost convergent.*

To see that a fails to be almost convergent, let p be arbitrarily large. Choose n to be the first index of a section of 1s of length p or greater. Then the quantity $\dfrac{a_n + \cdots + a_{n+p-1}}{p}$ equals 1. Similarly, letting n be the first index of a similarly long block of -1s gives $\dfrac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p}$ equal to -1. This shows that a fails the second condition of Theorem II.5 and is therefore not almost convergent.

Every almost summable sequence is Cesàro convergent. This fact comes from setting $n = 1$ in the second condition on Theorem II.5. But the converse is false, i.e. not every Cesàro convergent sequence is almost convergent. Take the sequence a from the previous example. This sequence was seen not to be almost convergent. Yet we can prove that it is Cesàro convergent.

We will use the following notation going forward. Given a number λ, Ces_λ will be the set of sequences whose Cesàro averages converge to λ and c_λ will be the set of sequences that converge to λ in the usual sense.

Lemma II.7. *The sequence* $a \in Ces_0$*.*

Proof. We formally define a in the following way. Let $n_k = \displaystyle\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$ for $k \geq 1$. Let $n_0 = 0$. For $k \in \mathbb{N}$ we define the kth block of a to be the part of the sequence starting with index $n_{k-1} + 1$ and ending with index n_k.

If k is odd, then we define all the terms in the kth block to 1. That is, if k is odd and $n_{k-1} < n \leq n_k$, then $a_n = 1$. Alternately, if k is even and $n_{k-1} < n \leq n_k$ we have $a_n = -1$.

Now define $b = C(a)$. We notice immediately that b is monotone on the blocks defined above. If $k > 1$ is odd, then

$$b_{n_{k-1}} < b_n < b_{n_k} \quad \forall n \in (n_{k-1}, n_k).$$

If $k > 1$ is even, then

$$b_{n_{k-1}} > b_n > b_{n_k} \quad \forall n \in (n_{k-1}, n_k).$$

This gives

$$\limsup_{n\to\infty} b_n = \limsup_{k\to\infty} b_{n_k} \qquad \text{and} \qquad \liminf_{n\to\infty} b_n = \liminf_{k\to\infty} b_{n_k}.$$

Thus, the Lemma will be proven if we can show $b_{n_k} \underset{k}{\to} 0$.

This is true and in fact $(b_{n_k})_k = (1, -1/3, 1/3, -1/5, 1/5, -1/7, 1/7, ...)$. Indeed, $b_{n_1} = b_1 = 1$ and $b_{n_2} = b_3 = \frac{1-1-1}{3} = \frac{-1}{3}$. We will check the rest by induction. Let $j \geq 1$ be given.

$$\begin{aligned}
b_{n_{2j+1}} &= \sum_{k=1}^{n_{2j+1}} \frac{a_k}{n_{2j+1}} = \sum_{k=1}^{n_{2j}} \frac{a_k}{n_{2j+1}} + \sum_{k=n_{2j}+1}^{n_{2j+1}} \frac{a_k}{n_{2j+1}} \\
&= \frac{n_{2j}}{n_{2j+1}} \sum_{k=1}^{n_{2j}} \frac{a_k}{n_{2j}} + \sum_{k=n_{2j}+1}^{n_{2j+1}} \frac{1}{n_{2j+1}} = \frac{n_{2j}}{n_{2j+1}} b_{n_{2j}} + \frac{2j+1}{n_{2j+1}} \\
&= \frac{2j(2j+1)}{(2j+1)(2j+2)} b_{n_{2j}} + \frac{(2j+1)2}{(2j+1)(2j+2)} = \frac{(2j)}{(2j+2)} \frac{(-1)}{(2j+1)} + \frac{1}{j+1} \\
&= \frac{1}{j+1}\left(\frac{-j}{2j+1} + 1\right) = \frac{1}{j+1}\left(\frac{j+1}{2j+1}\right) = \frac{1}{2j+1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_{n_{2j+2}} &= \sum_{k=1}^{n_{2j+2}} \frac{a_k}{n_{2j+2}} = \sum_{k=1}^{n_{2j+1}} \frac{a_k}{n_{2j+1}} \frac{n_{2j+1}}{n_{2j+2}} + \sum_{k=n_{2j+1}+1}^{n_{2j+2}} \frac{a_k}{n_{2j+2}} \\
&= b_{n_{2j+1}} \frac{n_{2j+1}}{n_{2j+2}} + (-1)\frac{2j+2}{n_{2j+2}} = \left(\frac{1}{2j+1}\right) \frac{2j+1}{2j+3} - \frac{(2j+2)\cdot 2}{(2j+2)(2j+3)} \\
&= \left(\frac{1}{2j+3}\right)(1-2) = \frac{-1}{2j+3}.
\end{aligned}$$

So $b_{n_{2j+1}} = \frac{1}{2j+1}$ and $b_{n_{2j}} = \frac{-1}{2j+1}$, giving $b_{n_j} \to 0$. $\square$

These preliminaries aside, our main interest in Banach limits is summarized by the following question. Given a summability method M, can we write down a Banach limit which recognizes M convergence? We will see that this question is interesting. That is, we have to be careful in our construction in order to guarantee this behavior. For example, we have just seen that $Ca \to 0$ and yet there are Banach limits which disagree on the value of a. The details of our constructions which address these inconsistencies will come after a discussion of iterates of certain (sequence-to-sequence) summability methods.

C. ITERATED CESÀRO AVERAGING OF BOUNDED SEQUENCES

Theorem II.8. *There is a bounded sequence* $x = (x_1, x_2, ...)$ *with* $\limsup x > \liminf x$ *such that*

$$\limsup x = \limsup Cx = \limsup C^2 x = \cdots = \limsup C^n x = \cdots$$

and

$$\liminf x = \liminf Cx = \liminf C^2 x = \cdots = \liminf C^n x = \cdots .$$

Proof. In this construction we will use the notation $C_n y$ to mean $\frac{1}{n}\sum_{k=1}^{n} y_k$ for any sequence $y = (y_1, y_2, ...)$.

Let $(\epsilon_n) \subseteq (0,1)$ be such that $\epsilon_n \underset{n}{\to} 0$. Let $x_1 = 1$. Note that $C_1 x = 1$.

By the regularity of C we can append a certain number, M_1, of zeros to x_1, i.e.

$$(x_1, x_2, ..., x_{M_1+1}) = (1, 0, ..., 0),$$

so that $C_{M_1+1}x < \epsilon_1$ and $C^2_{M_1+1}x < \epsilon_1$.

Continue constructing x by appending a certain number, M_2, of ones

$$(x_1, x_2, ..., x_{M_1+1}, x_{M_1+2}, ..., x_{M_2+M_1+1}) = (1, 0, ..., 0, 1, ..., 1)$$

so that every element of $\{C_{M_2+M_1+1}x, C^2_{M_2+M_1+1}x, C^3_{M_2+M_1+1}x\}$ is greater than $1-\epsilon_2$. Again, this is possible by the regularity of C.

Continue inductively. If j is even, then take the finite sequence $(x_1, x_2, ..., x_{M_j+M_{j-1}+\cdots+1})$ and append a certain number, M_{j+1}, of zeros so that every element of

$$\{C_{M_{j+1}+M_j+\cdots+1}x, C^2_{M_{j+1}+M_j+\cdots+1}x, ..., C^{j+2}_{M_{j+1}+M_j+\cdots+1}x\}$$

is less than ϵ_{j+1}.

If j is odd, then take the finite sequence $(x_1, x_2, ..., x_{M_j+M_{j-1}+\cdots+1})$ and append a certain number, M_{j+1}, of ones so that every element of

$$\{C_{M_{j+1}+M_j+\cdots+1}x, C^2_{M_{j+1}+M_j+\cdots+1}x, ..., C^{j+2}_{M_{j+1}+M_j+\cdots+1}x\}$$

is greater than $1 - \epsilon_{j+1}$.

This process generates a sequence $x = (1, 0, ..., 0, 1, ..., 1, 0, ...) \in \ell^\infty$. For every $m, k \in \mathbb{N}$, this sequence has the property that

$$\limsup_n C_n^m x \geq 1 - \epsilon_{m+k} \quad \text{and} \quad \liminf_n C_n^m x \leq \epsilon_{m+k}.$$

Therefore,

$$\limsup_n C_n^m x = 1 \quad \text{and} \quad \liminf_n C_n^m x = 0.$$

□

The following result greatly strengthens the above proposition in a way. We can show that if Cx fails to converge, then we can write down a lower bound on $\limsup C^2x - \liminf C^2x$ in terms of $\limsup Cx - \liminf Cx$. Working on the assumption that iterating the Cesàro operator would generate additional convergence, this result came as quite a surprise.

For all $s, t \in \mathbb{R}$, we define $s \vee t := \max\{s, t\}$. Also, c denotes the space of all convergent sequences.

Theorem II.9. *Let $x = (x_n)_{n \in \mathbb{N}} \in \ell^\infty$ with $y := Cx \notin c$. We define*

$$q := \limsup_{n\to\infty} \, x_n \quad \textit{and} \quad p := \liminf_{n\to\infty} \, x_n \ ;$$

and also

$$b := \limsup_{n\to\infty} \, y_n \quad \textit{and} \quad a := \liminf_{n\to\infty} \, y_n \ .$$

Let $d := b - a$ and $m := (a - p) \vee (q - b)$. Then

$$\limsup_{n\to\infty} \, (C^2x)_n - \liminf_{n\to\infty} \, (C^2x)_n \geq \frac{d^2}{10\,d + 8\,m + \sqrt{(10\,d + 8\,m)^2 - 4\,d^2}} \ . \tag{$\clubsuit$}$$

In particular, $z := C^2x \notin c$.

Moreover, in the special case where $b = q$ and $a = p$, we have that

$$\limsup_{n\to\infty} \, (C^2x)_n - \liminf_{n\to\infty} \, (C^2x)_n \geq \frac{d}{10 + \sqrt{96}} \ .$$

Proof. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$. Assume that $y := Cx \notin c$. For all $n \in \mathbb{N}$,

$$y_n := (Cx)_n := \frac{1}{n}\sum_{k=1}^{n} x_k \ .$$

Clearly,

$$-\infty < -\|x\|_\infty \le p \le a < b \le q \le \|x\|_\infty < \infty \ , \text{ and}$$

$$d := b - a > 0 \ .$$

We will show that $z := Cy = C^2x \notin c$ by proving that inequality (♣) holds.

Fix real numbers θ, τ, θ' and τ' with $0 < \theta < \tau < 1/2$ and $0 < \theta' < \tau' < 1/2$. Later we will see how to choose θ, τ, θ' and τ' in a certain optimal way, that enables us to achieve the right hand side of inequality (♣).

Fix $\epsilon \in \big(0, (1/2-\tau) \wedge (1/2-\tau')\big)$. So, $\tau < 1/2-\epsilon$ and $\tau' < 1/2-\epsilon$.

Then there exists $K_\epsilon \in \mathbb{N}$ with $K_\epsilon > 1/\epsilon$ such that for all $n \ge K_\epsilon$,

$$a - \epsilon\, d < (Cx)_n < b + \epsilon\, d \quad \text{and} \quad p - \epsilon\, d < x_n < q + \epsilon\, d \ .$$

Fix $N_0 \ge K_\epsilon$. There are two cases: $\big[$Case 1: $(C^2x)_{N_0} \le \frac{a+b}{2}\big]$; and $\big[$Case 2: $(C^2x)_{N_0} \ge \frac{a+b}{2}\big]$. Let's now consider Case 1.

Case 1: $(C^2x)_{N_0} \le \dfrac{a+b}{2}$.

Since $b = \limsup_{n\to\infty} (Cx)_n$, there exists $N_1 \in \mathbb{N}$ with $N_1 > N_0$ such that

$$(Cx)_{N_1} > b - \epsilon\, d = \frac{a+b}{2} + \left(\frac{1}{2} - \epsilon\right) d \ .$$

Recall that $\tau \in (0, 1/2-\epsilon)$. Also, $a = \liminf_{n\to\infty} (Cx)_n$. Therefore, there exists $N_2 \in \mathbb{N}$ with $N_2 > N_1$ such that

$$(Cx)_{N_2} < \frac{a+b}{2} + \tau\, d \ .$$

Further, we may assume that N_2 is as small as possible with this property. Hence,

$$(Cx)_n \ge \frac{a+b}{2} + \tau\, d \ , \text{ for all } n \in \{N_1, \dots, N_2 - 1\} \ . \tag{†}$$

Let $J := N_2 - N_1 \in \mathbb{N}$. Then,

$$\begin{aligned}
\frac{a+b}{2} + \tau\, d &> (Cx)_{N_2} = (Cx)_{N_1+J} \\
&= \frac{1}{N_1+J} \sum_{k=1}^{N_1+J} x_k = \frac{1}{N_1+J} \sum_{k=1}^{N_1} x_k + \frac{1}{N_1+J} \sum_{k=N_1+1}^{N_1+J} x_k \\
&\geq \frac{N_1}{N_1+J} (Cx)_{N_1} + \frac{J}{N_1+J} (p - \epsilon\, d) \\
&\geq \frac{N_1}{N_1+J} (b - \epsilon\, d) + \frac{J}{N_1+J} (p - \epsilon\, d)\ .
\end{aligned}$$

Therefore,

$$(N_1 + J) \left(\frac{a+b}{2} + \tau\, d \right) \geq N_1\, (b - \epsilon\, d) + J\, (p - \epsilon\, d)\ ;$$

and so,

$$J \left(\frac{a+b}{2} + \tau\, d - (p - \epsilon\, d) \right) \geq N_1 \left[(b - \epsilon\, d) - \left(\frac{a+b}{2} + \tau\, d \right) \right]\ .$$

Thus,

$$J \left[\left(\frac{1}{2} + \tau + \epsilon \right) d + a - p \right] \geq N_1 \left(\frac{1}{2} - \tau - \epsilon \right) d\ ;$$

and consequently,

$$J \geq \frac{N_1\, \left(\frac{1}{2} - \epsilon - \tau \right)}{\left[\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d} \right]}\ . \tag{$\bigstar$}$$

From above, we have that $\theta \in (0, \tau)$.

Sub-case 1.a: $(C^2 x)_{N_1} \geq \dfrac{a+b}{2} + \theta\, d$. Recall that $(C^2 x)_{N_0} \leq \dfrac{a+b}{2}$. So,

$$(C^2 x)_{N_1} - (C^2 x)_{N_0} \geq \theta\, d\ .$$

Sub-case 1.b: $(C^2 x)_{N_1} < \dfrac{a+b}{2} + \theta\, d$.

Then, by (†) above,

$$\begin{aligned}
&(C^2x)_{N_1+J-1} - (C^2x)_{N_1} \\
= \; & \frac{1}{N_1+J-1} \sum_{k=1}^{N_1+J-1} (Cx)_k - (C^2x)_{N_1} \\
= \; & \frac{1}{N_1+J-1} \sum_{k=1}^{N_1} (Cx)_k + \frac{1}{N_1+J-1} \sum_{k=N_1+1}^{N_1+J-1} (Cx)_k - (C^2x)_{N_1} \\
\geq \; & \frac{N_1}{N_1+J-1} (C^2x)_{N_1} + \frac{J-1}{N_1+J-1} \left(\frac{a+b}{2} + \tau\, d \right) - (C^2x)_{N_1} \\
\geq \; & \frac{J-1}{N_1+J-1} \left(\frac{a+b}{2} + \tau\, d \right) - \frac{J-1}{N_1+J-1} (C^2x)_{N_1} \\
\geq \; & \frac{J-1}{N_1+J-1} \left(\frac{a+b}{2} + \tau\, d \right) - \frac{J-1}{N_1+J-1} \left(\frac{a+b}{2} + \theta\, d \right) \\
= \; & \left(1 - \frac{N_1}{N_1+J-1} \right) (\tau - \theta)\, d \ .
\end{aligned}$$

From Fact (★) above,

$$\begin{aligned}
(C^2x)_{N_1+J-1} - (C^2x)_{N_1} \;&\geq\; \left(1 - \frac{N_1}{N_1 + \left(\frac{N_1 \left(\frac{1}{2} - \epsilon - \tau \right)}{\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d}} \right) - 1} \right) (\tau - \theta)\, d \\
&=\; \left(1 - \frac{1}{1 + \left(\frac{\frac{1}{2} - \epsilon - \tau}{\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d}} \right) - \frac{1}{N_1}} \right) (\tau - \theta)\, d \\
&\geq\; \left(1 - \frac{1}{1 + \left(\frac{\frac{1}{2} - \epsilon - \tau}{\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d}} \right) - \epsilon} \right) (\tau - \theta)\, d \ .
\end{aligned}$$

Note that this last inequality is valid because $1/N_1 < \epsilon$ and

$$1 + \left(\frac{\frac{1}{2} - \epsilon - \tau}{\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d}} \right) - \epsilon = \frac{1 + \frac{a-p}{d}}{\frac{1}{2} + \epsilon + \tau} - \epsilon > \frac{1 + \frac{a-p}{d}}{\frac{1}{2} + \epsilon + \frac{1}{2} - \epsilon} - \epsilon \geq 1 - \epsilon > 0 \ .$$

It follows from Sub-case 1.a and Sub-case 1.b that

$$\sup_{N \geq N_0} (C^2x)_N - \inf_{M \geq N_0} (C^2x)_M \geq U_\epsilon \ ,$$

where

$$U_\epsilon := \min\left\{\theta\, d, \left(1 - \frac{1}{1+\left(\frac{\frac{1}{2}-\epsilon-\tau}{\frac{1}{2}+\epsilon+\tau+\frac{a-p}{d}}\right)-\epsilon}\right)(\tau-\theta)\, d\right\}\ . \tag{$\heartsuit$}$$

Case 2: $(C^2x)_{N_0} \geq \dfrac{a+b}{2}$.

Then $\big(C^2(-x)\big)_{N_0} \leq \big((-b)+(-a)\big)/2$. From Case 1, with θ replaced by θ', τ replaced by τ', $-b$ instead of a, $-a$ instead of b, and $-q$ instead of p, we get that

$$\sup_{N\geq N_0}(C^2x)_N - \inf_{M\geq N_0}(C^2x)_M = \sup_{M\geq N_0}\big(C^2(-x)\big)_M - \inf_{N\geq N_0}\big(C^2(-x)\big)_N \geq U'_\epsilon\ ,$$

where

$$U'_\epsilon := \min\left\{\theta'\, d, \left(1 - \frac{1}{1+\left(\frac{\frac{1}{2}-\epsilon-\tau'}{\frac{1}{2}+\epsilon+\tau'+\frac{-b-(-q)}{d}}\right)-\epsilon}\right)(\tau'-\theta')\, d\right\}\ . \tag{$\diamondsuit$}$$

It follows from Case 1 and Case 2 that for all integers $N_0 \geq K_\epsilon$,

$$\sup_{N\geq N_0}(C^2x)_N - \inf_{M\geq N_0}(C^2x)_M \geq U_\epsilon \wedge U'_\epsilon\ .$$

Consequently, letting $N_0 \to \infty$, we see that

$$\limsup_{n\to\infty}(C^2x)_n - \liminf_{n\to\infty}(C^2x)_n \geq U_\epsilon \wedge U'_\epsilon\ .$$

But $\epsilon \in \big(0, (1/2-\tau)\wedge(1/2-\tau')\big)$ is arbitrary. Letting $\epsilon \to 0+$ in ($\heartsuit$) and ($\diamondsuit$) above, we see that

$$\limsup_{n\to\infty}(C^2x)_n - \liminf_{n\to\infty}(C^2x)_n \geq U \wedge U'\ ,$$

where

$$\begin{aligned} U &= \min\left\{\theta\, d, \left(1 - \frac{1}{1+\left(\frac{\frac{1}{2}-\tau}{\frac{1}{2}+\tau+\frac{a-p}{d}}\right)}\right)(\tau-\theta)\, d\right\} \\ &= \min\left\{\theta, \frac{\left(\frac{1}{2}-\tau\right)}{1+\frac{a-p}{d}}(\tau-\theta)\right\} d\ , \end{aligned}$$

and

$$U' = \min\left\{\theta', \frac{\left(\frac{1}{2}-\tau'\right)}{1+\frac{q-b}{d}}\,(\tau'-\theta')\right\}\,d\ .$$

Here, recall that the real numbers θ, τ, θ' and τ' with $0<\theta<\tau<1/2$ and $0<\theta'<\tau'<1/2$ are arbitrary.

Fix $\theta\in(0,1/2)$. Consider positive real numbers u and v such that $u+v=1/2-\theta$; e.g., $u:=1/2-\tau$ and $v=\tau-\theta$, when $\tau\in(\theta,1/2)$. Then the product $u\,v$ is *maximal* when $u=v=(1/2)\,(1/2-\theta)$.

Thus, when $\tau\in(\theta,1/2)$, the product $(1/2-\tau)\,(\tau-\theta)$ is maximal when $1/2-\tau=\tau-\theta=(1/2)\,(1/2-\theta)$; i.e.,

$$\tau=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}-\theta\right)=\frac{1}{4}+\frac{\theta}{2}\ .$$

We may argue similarly using θ' and τ'. So, for all $\theta,\theta'\in(0,1/2)$,

$$\limsup_{n\to\infty}\,(C^2x)_n-\liminf_{n\to\infty}\,(C^2x)_n\geq W_\theta\wedge W'_{\theta'}\ ;$$

where

$$W_\theta=\min\left\{\theta, \frac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}\right\}\,d\quad,\quad\text{and}\quad W'_{\theta'}=\min\left\{\theta', \frac{\frac{1}{4}\left(\frac{1}{2}-\theta'\right)^2}{1+\frac{q-b}{d}}\right\}\,d\ .$$

Note that $\theta\mapsto\theta$ is a *strictly increasing* function on the interval $(0,1/2)$, and $\theta\mapsto(1/4)\,(1/2-\theta)^2\big/(1+(a-p)/d)$ is a *strictly decreasing* function on $(0,1/2)$. Therefore, the function $\theta\mapsto\min\big\{\theta,(1/4)\,(1/2-\theta)^2\big/(1+(a-p)/d)\big\}$ is *maximal* on $(0,1/2)$ precisely when

$$\theta=\frac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}\ .$$

Solving this quadratic equation for $\theta\in(0,1/2)$ yields:

$$\theta=\frac{1}{2}\left[5+4\left(\frac{a-p}{d}\right)\right]-\frac{1}{2}\sqrt{\left[5+4\left(\frac{a-p}{d}\right)\right]^2-1}\ .$$

From this, and a similar argument involving θ', we conclude that

$$\limsup_{n\to\infty}\,(C^2x)_n-\liminf_{n\to\infty}\,(C^2x)_n\geq(\Theta\wedge\Theta')\,d\ ;$$

where

$$\Theta := \frac{1}{2}\left[5+4\left(\frac{a-p}{d}\right)\right] - \frac{1}{2}\sqrt{\left[5+4\left(\frac{a-p}{d}\right)\right]^2 - 1}\,,$$

and

$$\Theta' := \frac{1}{2}\left[5+4\left(\frac{q-b}{d}\right)\right] - \frac{1}{2}\sqrt{\left[5+4\left(\frac{q-b}{d}\right)\right]^2 - 1}\,.$$

Let $m := (a-p) \vee (q-b) = \max\{a-p, q-b\}$. It follows that

$$\begin{aligned}(\Theta \wedge \Theta')\, d &= \frac{d/4}{\frac{1}{2}\left[5+\frac{4m}{d}\right] + \frac{1}{2}\sqrt{\left[5+\frac{4m}{d}\right]^2 - 1}} \\ &= \frac{d^2}{10\,d + 8\,m + \sqrt{(10\,d+8\,m)^2 - 4\,d^2}}\,.\end{aligned}$$

Finally, in the special case where $b = q$ and $a = p$ (so that $m = 0$), we see that

$$\limsup_{n\to\infty}\,(C^2x)_n - \liminf_{n\to\infty}\,(C^2x)_n \geq \frac{d}{10+\sqrt{96}}\,.$$

□

Open Question II.10. *Can the bounds in the previous theorem be improved?*

Question II.11. *Are there versions of Theorem II.9 corresponding to other regular summability methods?*

It is easier than the construction in Theorem II.8 to see that there exist bounded sequences x of real numbers so that the corresponding sequence of Cesàro iterates, Cx, fails to converge. For any such sequence, x, we have from the above theorem that the n-th Cesàro iterate $C^n x$ also fails to converge for every integer $n \geq 2$.

The qualitative version of this theorem: [for bounded sequences x, Cx is convergent if and only if C^2x is convergent] follows from the Abelian theorem that Abel convergence is stronger than Cesàro convergence (this is due to Frobenius [26]) and a classical and deep Tauberian theorem of Hardy and Littlewood which follows (see e.g. Thm 7.3 of [33] for details).

Theorem II.12. *If (x_n) is bounded and Abel-convergent, then (x_n) is Cesàro convergent.*

But we can also avoid Abel convergence by using the following Theorem of Hardy from 1910 [29].

Theorem II.13. *If (x_n) is Cesàro convergent to L and $x_{n+1} - x_n = \mathcal{O}(\frac{1}{n})$, then $x_n \to L$.*

And now we have our qualitative result as a Corollary.

Corollary II.14. *If $x \in \ell^\infty$ and C^2x converges, then Cx converges.*

Proof. The assumption that $C^2x = C(Cx)$ converges means that Cx is Cesàro convergent. If we can show that $C_{n+1}(x) - C_n(x)$ is $\mathcal{O}(\frac{1}{n})$, then Theorem II.13 will finish the proof.

$$\begin{aligned} |C_{n+1}(x) - C_n(x)| &= \left|\sum_{k=1}^{n+1} \frac{x_k}{n+1} - \sum_{k=1}^{n} \frac{x_k}{n}\right| \\ &= \left|\frac{x_{n+1}}{n+1} + \sum_{k=1}^{n} x_k \left(\frac{1}{n+1} - \frac{1}{n}\right)\right| \\ &\leq \|x\|_\infty \left(\frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{n(n+1)}\right) = \|x\|_\infty \frac{2}{n+1}. \end{aligned}$$

□

Open Question II.15. *Which (non-Cesàro convergent) $x \in \ell^\infty$ are such that*

$$\limsup C(C(x)) - \liminf C(C(x)) = \limsup C(x) - \liminf C(x)?$$

We have just proven (Theorem II.8) that there are some sequences so that this happens. It also seems, based on some preliminary numerical work, that there are some sequences so that this does not happen.

D. TRANSLATED CESÀRO AVERAGING

A variation of Cesàro averaging is the *Translated Cesàro Average*, Gz, of any scalar sequence z. This is defined to be the sequence for which the n-th term is

$$(Gz)_n := \frac{1}{n} \sum_{k=n}^{2n-1} z_k \ .$$

We can use G is a way that we can not use C. But first, note that we can prove the following Tauberian theorems for C and G.

Theorem II.16. *Let z be a bounded sequence of scalars and $\lambda \in \mathbb{K}$. Then Cz converges to λ if and only if Gz converges to λ.*

Proof. **$\Rightarrow$:** We will show that $\limsup_n |(Gz)_n - \lambda| = 0$.

$$\begin{aligned}
|(Gz)_n - \lambda| &= \left| \sum_{k=n}^{2n-1} \frac{z_k}{n} - \lambda \right| \\
&= \left| \sum_{k=n}^{2n-1} \frac{z_k}{n} + \sum_{k=1}^{n-1} \frac{z_k}{2n-1} - \sum_{k=1}^{n-1} \frac{z_k}{2n-1} - \lambda \right| \\
&= \left| \frac{2n-1}{n} \sum_{k=n}^{2n-1} \frac{z_k}{2n-1} + \sum_{k=1}^{n-1} \frac{z_k}{2n-1} - \sum_{k=1}^{n-1} \frac{z_k}{2n-1} - \lambda \right| \\
&= \left| \sum_{k=1}^{2n-1} \frac{z_k}{2n-1} + \left(1 - \frac{1}{n}\right) \sum_{k=n}^{2n-1} \frac{z_k}{2n-1} - \sum_{k=1}^{n-1} \frac{z_k}{2n-1} - \lambda \right| \\
&\le \left| \sum_{k=1}^{2n-1} \frac{z_k}{2n-1} - \lambda \right| + \left(\frac{n-1}{2n-1}\right) \left| \sum_{k=n}^{2n-1} \frac{z_k}{n} - \frac{n}{n-1} \sum_{k=1}^{n-1} \frac{z_k}{n} \right|.
\end{aligned}$$

In this last line, the left term is $|(Cz)_{2n-1} - \lambda|$, which converges to 0 by assumption. The right hand term is

$$\left(\frac{n-1}{2n-1}\right) |(Gz)_n - (Cz)_{n-1}|.$$

Because $Cz \to \lambda$, the lim sup of this quantity is exactly $1/2 \limsup |(Gz)_n - \lambda|$. This all implies that

$$\limsup |(Gz)_n - \lambda| \le \frac{1}{2} \limsup |(Gz)_n - \lambda|,$$

giving $Gz \to \lambda$.

$\Leftarrow$: Note that it suffices to show that $(Cz)_{2n-1} \to \lambda$ because

$$\left| \sum_{k=1}^{2n} \frac{z_k}{2n} - \sum_{k=1}^{2n-1} \frac{z_k}{2n-1} \right| = \left| \frac{z_{2n}}{2n} - \sum_{k=1}^{2n-1} \frac{z_k}{2n(2n-1)} \right| \le \frac{\|z\|_\infty}{n}.$$

Now we compute

$$\begin{aligned}
|(Cz)_{2n-1} - \lambda| &= \left| \sum_{k=1}^{2n-1} \frac{z_k}{2n-1} - \lambda \right| \\
&= \left| \frac{n-1}{2n-1} \sum_{k=1}^{n-1} \frac{z_k}{n-1} - \frac{n-1}{2n-1}\lambda + \frac{n}{2n-1} \sum_{k=n}^{2n-1} \frac{z_k}{n} - \frac{n}{2n-1}\lambda \right| \\
&\leq \frac{n-1}{2n-1}|(Cz)_{n-1} - \lambda| + \frac{n}{2n-1}|(Gz)_n - \lambda|.
\end{aligned}$$

Because $Gz \to \lambda$, using the same argument as for the converse we get

$$\limsup |(Cz)_{2n-1} - \lambda| \leq \frac{1}{2} \limsup |(Cz)_{n-1} - \lambda|,$$

which gives $Cz \to \lambda$. $\square$

E. ITERATED TRANSLATED CESÀRO AVERAGING OF BOUNDED SEQUENCES

If we consider a sequence $x = (x_1, x_2, ...)$ and the Cesàro map as previously defined, then we can construct an infinite matrix whose rows are the succesive iterates of the Cesàro map acting on x. Some numbers below are approximate so that the table will fit on the page.

$$\left|\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & \frac{1}{2}(x_1 + x_2) & \frac{1}{3}(x_1 + x_2 + x_3) & \frac{1}{4}(x_1 + \cdots + x_4) & \frac{1}{5}(x_1 + \cdots + x_5) \\
x_1 & \frac{3}{4}x_1 + \frac{1}{4}x_2 & \frac{11}{18}x_1 + \frac{5}{18}x_2 + \frac{1}{9}x_3 & \frac{25}{48}x_1 + \cdots + \frac{1}{16}x_4 & \frac{137}{300}x_1 + \cdots \\
x_1 & \frac{7}{8}x_1 + \frac{1}{8}x_2 & \frac{85}{108}x_1 + \frac{19}{108}x_2 + \frac{1}{27}x_3 & \frac{415}{576}x_1 + \cdots + \frac{1}{64}x_4 & \frac{2952}{4421}x_1 + \cdots \\
x_1 & \frac{15}{16}x_1 + \frac{1}{16}x_2 & \frac{575}{648}x_1 + \frac{65}{648}x_2 + \frac{1}{81}x_3 & \frac{2487}{2941}x_1 + \cdots + \frac{1}{256}x_4 & \frac{661}{816}x_1 + \cdots \\
x_1 & \frac{31}{32}x_1 + \frac{1}{32}x_2 & \frac{3661}{3888}x_1 + \frac{211}{3888}x_2 + \frac{1}{243}x_3 & \frac{1927}{2100}x_1 + \cdots + \frac{1}{1024}x_4 & \frac{9755}{10886}x_1 + \cdots \\
x_1 & \frac{63}{64}x_1 + \frac{1}{64}x_2 & \frac{2630}{2711}x_1 + \frac{138}{4841}x_2 + \frac{1}{729}x_3 & \frac{1491}{1558}x_1 + \cdots + \frac{1}{4096}x_4 & \frac{2671}{2827}x_1 + \cdots \\
x_1 & \frac{127}{128}x_1 + \frac{1}{128}x_2 & \frac{5519}{5604}x_1 + \frac{140}{9517}x_2 + \frac{1}{2187}x_3 & \frac{2961}{3028}x_1 + \cdots + \frac{1}{4^7}x_4 & \frac{2501}{2575}x_1 + \cdots \\
x_1 & \frac{254}{256}x_1 + \frac{1}{256}x_2 & \frac{1425}{1436}x_1 + \frac{35}{4662}x_2 + \frac{1}{6561}x_3 & \frac{2455}{2483}x_1 + \cdots + \frac{1}{4^8}x_4 & \frac{2068}{2099}x_1 + \cdots
\end{array}\right|$$

It was proven in [24] that by moving down any column of this matrix the limit will be x_1, which is not useful information about the sequence x. Alternately, by moving along row $n+1$ we get convergence if and only if $C^n(x)$ converges (which again converges if and only if Cx converges). So neither of these approaches generates a Banach limit which captures anything more than Cx. What about taking the limit down the diagonal? This does not work either. The reason is that the diagonal entries put too much weight onto x_1. We will prove this after establishing a notational convention.

Notice that every entry in the Cesàro matrix is of the form

$$C_j^n(x) = \sum_{k=1}^{j} \alpha_{n,j,k} x_k$$

where each $\alpha_{n,j,k} \geq 0$ and for any fixed n and j we have $\sum_{k=1}^{j} \alpha_{n,j,k} = 1$, and the $\alpha_{n,j,k}$ are independent of x.

We have the following formula which comes from the definition of the Cesàro map

$$\alpha_{n,j,k} = \sum_{i=1}^{j} \frac{\alpha_{n-1,i,k}}{j}.$$

Lemma II.17. *Whenever $n \geq j$ we have $\alpha_{n,j,1} \geq 1/2$, in particular $\alpha_{n,n,1} \geq 1/2$ for every $n \in \mathbb{N}$.*

Proof. By induction. $\alpha_{1,1,1} = 1$. In fact, $\alpha_{n,1,1} = 1$ for all n.

Now fix $n \in \mathbb{N}$. Suppose by way of induction that $\alpha_{n-1,j,1} \geq 1/2$ for all $j \leq n-1$. Now we'll consider any entry in the nth row up to the diagonal, i.e. $j \leq n$.

$$\begin{aligned} \alpha_{n,j,1} &= \sum_{i=1}^{j} \frac{\alpha_{n-1,i,1}}{j} = \frac{\alpha_{n-1,1,1}}{j} + \sum_{i=2}^{j-1} \frac{\alpha_{n-1,i,1}}{j} + \frac{\alpha_{n-1,j,1}}{j} \\ &\geq \frac{1}{j} + \sum_{i=2}^{j-1} \frac{1/2}{j} + \frac{0}{j} = \frac{1}{2}. \end{aligned}$$

$\square$

Note that $\forall x \in \ell^\infty$, $G(x) \in \ell^\infty$ with $\|G(x)\|_\infty \leq \|x\|_\infty$.

So for each $\nu \in \mathbb{N}$ we can define $x^\nu \in \ell^\infty$ recursively by

$$x^\nu = G(x^{\nu-1}) = \left(\frac{1}{n}\sum_{j=n}^{2n-1} x_j^{\nu-1}\right)_{n\in\mathbb{N}} .$$

It follows from the regularity of G that if x^ν converges to L, then $\forall \mu > \nu$, x^μ converges to L. Furthermore, the convergence gets no slower as μ increases. Some details are included in the proof of claim 3 below, where this bit about "no slower" needs to be made precise.

We are going to construct a Banach limit that recognizes G convergence and also the convergence of the iterates of G. It may be that (in light of the results above about the iterates of C) iterating G generates no additional convergence. Either way, this construction serves as an example of a detailed B-lim construction using the Hahn-Banach extension theorem. Also, this construction is designed to recognize the convergence down the diagonal of the G matrix (again, this is impossible for C because of the weight on x_1 in the C matrix). In fact, let us define a new method that represents the diagonal of the G matrix.

Definition II.18. *Let z be a scalar sequence. Define the scalar sequence $G^{\omega_0}z$ by setting its n-th coordinate to be*

$$\left(G^{\omega_0}z\right)_n := \left(G^n z\right)_n \text{ , for all } n \in \mathbb{N} \text{ .}$$

Define $V_n := \{x \in \ell^\infty : x^n \in c_\lambda, \lambda \in \mathbb{R}\}$. The above fact gives $V_n \subseteq V_{n+1}$. Define $M = \bigcup_{n\in\mathbb{N}} V_n$. Note that it is plausible that $M = V_1$.

Claim 1: M is a linear subspace of ℓ^∞.

Proof. If $x, y \in M$, then $x^n \to \lambda_x$, $y^m \to \lambda_y$, and the same holds for x^{n+m} and y^{n+m}. Then

$$G(x+y) = \left(\frac{1}{k}\sum_{j=n}^{2n-1}(x_j + y_j)\right)_{k\in\mathbb{N}}$$

is additive, giving

$$(x+y)^{n+m} = x^{n+m} + y^{n+m} \to \lambda_x + \lambda_y.$$

Similary,

$$G(\alpha x) = \left(\frac{1}{k}\sum_{j=n}^{2n-1} \alpha x_j\right)_{k\in\mathbb{N}} = \left(\frac{\alpha}{k}\sum_{j=n}^{2n-1} x_j\right)_{k\in\mathbb{N}} = \alpha G(x)$$

so when $x^n \to \lambda_x$, we get $\alpha x^n \to \alpha\lambda_x$. $\square$

By the definition of M we have that $\forall x \in M$, $\exists n \in \mathbb{N}, \lambda \in \mathbb{R}$ such that $x^n \in c_\lambda$. In this case define $f(x) := \lambda$.

Claim 2: $f : M \to \mathbb{R}$ is linear.

Proof. The proof of Claim 1 gives that $\forall x, y \in M$, $f(x+y) = f(x) + f(y)$, and that $f(\alpha x) = \alpha f(x)$. $\square$

Define $\rho : \ell^\infty \to \mathbb{R}$ by $\rho(x) = \limsup\{x_n^n : n \in \mathbb{N}\}$. The following claim is where the speed of convergence of $G(x)$ is important, and illustrates why regular Cesàro averaging does not work for this construction.

Claim 3: $\rho(x) \geq f(x)$ $\forall x \in M$.

Proof. If $x \in M$, then x^ν converges to λ for some $\lambda \in \mathbb{R}, \nu \in \mathbb{N}$. This means that $\forall \epsilon > 0$, $\exists N(\epsilon)$ s.t. $\forall n > N(\epsilon)$ we have $|x_n^\nu - \lambda| < \epsilon$.

Now consider the sequence $x^{\nu+1} = G(x^\nu)$. Fix $\epsilon_0 > 0$, and take $N(\epsilon_0)$ to be as given in the previous paragraph. Then for any $n > N(\epsilon_0)$ we have

$$|x_n^{\nu+1} - \lambda| = \left|\left(\sum_{k=n}^{2n-1} \frac{1}{n} x_k^\nu\right) - \lambda\right| = \left|\sum_{k=n}^{2n-1} \frac{x_k^\nu - \lambda}{n}\right|$$

$$\leq \sum_{k=n}^{2n-1} \frac{|x_k^\nu - \lambda|}{n} \leq \sum_{k=n}^{2n-1} \frac{\epsilon_0}{n} = \epsilon_0.$$

Then by induction, we can say that $\forall j > \nu$, $\forall \epsilon > 0$, $\forall n > N(\epsilon)$, we have

$$|x_n^j - \lambda| < \epsilon.$$

In particular, $\forall \epsilon > 0$, $\forall n > \max\{\nu, N(\epsilon)\}$ we have $|x_n^n - \lambda| < \epsilon$, which means that

$$\lim_n x_n^n = \lambda \Rightarrow \rho(x) = \limsup_n x_n^n = \lambda = f(x).$$

□

Claim 4: ρ is a sublinear functional.

Proof. For any $x, y \in \ell^\infty$, the fact that $\rho(x+y) \leq \rho(x) + \rho(y)$ comes from applying a well-known fact about lim sup to the sequences $\{x_n^n\}$, $\{y_n^n\}$, $\{x_n^n + y_n^n\}$.

Now let $\alpha \geq 0$ be given. In the proof of Claim 1 it was noted that $G(\alpha x) = \alpha G(x)$. Inductively, we see that $\forall n \in \mathbb{N}$,

$$(\alpha x)^n = \alpha G^n(x) = \alpha(x^n).$$

Then we have (keeping in mind that $\alpha \geq 0$)

$$\rho(\alpha x) = \limsup_n (\alpha x)_n^n = \limsup_n \alpha(x_n^n) = \alpha \limsup_n x_n^n = \alpha\, \rho(x).$$

□

Now we can use the Hahn-Banach Extension Theorem to generate our Banach Limit.

Theorem II.19. ***[Hahn-Banach Extension Theorem]*** *Let $(X, \|\cdot\|)$ be a Banach space. Suppose that $\rho : X \to \mathbb{R}$ is sublinear. Let M be a subspace of X. If $f : M \to \mathbb{R}$ is linear such that $f(x) \leq \rho(x)$ for all $x \in M$, then there exists a linear $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in M$ and $F(x) \leq \rho(x)$ for all $x \in X$.*

Claim 5: The linear $F : \ell^\infty \to \mathbb{R}$ guaranteed by the Hahn-Banach Extension Theorem is shift invariant.

Proof. Consider the function $S : (x_1, x_2, x_3, x_4, ...) \to (x_2, x_3, x_4, x_5, ...)$. Now, for any $x \in \ell^\infty$ consider the vector $y = x - S(x)$. This vector is $(x_1 - x_2, x_2 - x_3, ..., x_n - x_{n+1}, ...)$. We will show that $G(y)$ converges to 0.

The nth coordinate of $G(y)$ is

$$\sum_{k=n}^{2n-1} \frac{1}{n}(x_k - x_{k+1}) = \frac{x_n - x_{2n}}{n}.$$

Finally,

$$\left| \frac{x_n - x_{2n}}{n} \right| \leq \frac{2\|x\|_\infty}{n} \underset{n}{\to} 0.$$

This means that $0 = F(y) = F(x - S(x)) = F(x) - F(S(x)) \Rightarrow F(x) = F(S(x))$. □

Open Question II.20. *This Banach limit F is such that if $n \in \mathbb{N}$, $z \in \ell^\infty$, $\lambda \in \mathbb{K}$ and $G^n z$ converges to λ, then $Fz = \lambda$. Is it also true that if $G^{\omega_0} z$ converges to λ, then $Fz = \lambda$?*

Open Question II.21. *Can we extend this construction to larger ordinals; and if so, how far?*

F. BANACH LIMITS THAT ARE INVARIANT UNDER A VARIETY OF OPERATORS

In light of Theorem II.16, this Banach limit F from the previous section satisfies the condition: $\left[F(x) = \lim_{n\to\infty} (Cx)_n, \text{ for all } x \in Ces \right]$. We have seen that it is not the case that *all* Banach limits σ have the property that $\left[\sigma(x) = \lim_{n\to\infty} (Cx)_n, \text{ for all } x \in Ces \right]$.

So F is especially well-behaved in this sense. We will try to do even better by addressing the following question.

Question II.22. *Does there exist a Banach limit σ such that*

$$\sigma(x) = \sigma(Cx) \ , \ \textit{for all } x \in \ell^\infty \ ?$$

Question II.22 was answered affirmatively in [18]. Their proof uses the Markov-Kakutani fixed point theorem. Our proof below uses the following version of the Hahn-Banach extension theorem from Royden's *Real Analysis* [46]. Further developments of this idea can be found in [50].

Theorem II.23. *(Proposition 5, page 224 of [46]) Let X be a vector space and let V be a subspace of X. Let $p : X \to \mathbb{R}$ be subadditive and positive homogeneous. Let $f : V \to \mathbb{R}$ be linear with the property that $f(v) \leq p(v)$ $\forall v \in V$.*

Let G be an Abelian semigroup of linear operators on X such that for every $A \in G$ we have $p(Ax) \leq p(x)$ $\forall x \in X$, while for every $v \in V$ we have $Av \in V$ and $f(Av) = f(v)$. Then there is an extension F of f to a linear functional on X such that $F(x) \leq p(x)$ and $F(Ax) = F(x)$ for all $x \in X$.

We are going to use this theorem to answer question II.22. If we try to let $X = \ell^\infty$ and let G be generated by S (the left-shift operator) and C, we encounter a problem. For example, let $x = (1, 0, 0, 0, ...)$. Then $Sx = (0, 0, 0, ...)$. So $CSx = (0, 0, 0, ...)$ while $SCx = (1/2, 1/3, 1/4, ...)$. So G would not be Abelian on X.

Our solution will be to mod out by c_0. Note that if two sequences $x, y \in \ell^\infty$ differ by a sequence $z \in c_0$, then x converges to λ (resp. Cesàro converges to λ) if and only if y converges to λ (resp. Cesàro converges to λ). So, passing to c_0 equivalence classes, we will lose no information about limits.

Lemma II.24. *For every $x \in \ell^\infty$, $SCx - CSx \in c_0$.*

Proof. Fix an $x = (x_1, x_2, ...) \in \ell^\infty$.

$$\begin{aligned} |(SCx)_n - (CSx)_n| &= \left| \frac{x_1 + \cdots + x_{n+1}}{n+1} - \frac{x_2 + \cdots + x_{n+1}}{n} \right| \\ &= \left| \frac{x_1}{n+1} - \frac{x_2}{n(n+1)} - \cdots - \frac{x_{n+1}}{n(n+1)} \right| \\ &\leq \frac{\|x\|_\infty}{n+1} + \sum_{j=2}^{n+1} \frac{\|x\|_\infty}{n(n+1)} = 2\frac{\|x\|_\infty}{n+1}. \end{aligned}$$

Because this converges to zero as $n \to \infty$ we have $SCx - CSx \in c_0$. □

Lemma II.25. *If $x \in \ell^\infty$, $n \in \mathbb{N}$, $A = S$ or $A = C$, and for every $j \in \{1, ..., n\}$ we have $B_j = S$ or $B_j = C$, then $AB_1 \cdots B_n x - B_1 \cdots B_n Ax \in c_0$.*

Proof. Fix $x \in \ell^\infty$. We will proceed by induction on n. Lemma II.24 (and the fact that C commutes with itself) establishes the claim for $n = 1$.

Now let $k > 1$ be given. Suppose that the claim holds for $n = k - 1$. For every $j \in \{1, ..., k\}$ let B_j be chosen from $\{S, C\}$. We want to show that $Dx = AB_1 \cdots B_n x - B_1 \cdots B_n Ax$ is in c_0. Consider that

$$Dx = AB_1 \cdots B_n x - B_1 AB_2 \cdots B_n x + B_1 AB_2 \cdots B_n x - B_1 B_2 \cdots B_n Ax.$$

Now, $y = B_2 \cdots B_n x$ is in ℓ^∞. So we can re-write the first difference above as $AB_1 y - B_1 Ay$, which is in c_0 by Lemma II.24. The second difference is

$$B_1(AB_2 \cdots B_n x - B_2 \cdots B_n Ax).$$

We know that $AB_2 \cdots B_n x - B_2 \cdots B_n Ax$ is in c_0 by the inductive hypothesis. Since B_1 is regular, it follows that $B_1(AB_2 \cdots B_n x - B_2 \cdots B_n Ax)$ is also in c_0. This means Dx is the sum of two elements of c_0 and therefore also in c_0. $\square$

Now we have that all compositions of S and C commute with each other in $X = \ell^\infty / c_0$. That is, let G_0 be the semigroup of linear operators on ℓ^∞ generated by $\{I, S, C\}$. Let G be the associated semigroup acting on X. Formally, for every $T_0 \in G_0$ we define $T \in G$ by letting $T([x]) = [T(x)]$ (where $x \in \ell^\infty$ and $[x]$ is understood to be the equivalence class containing x in ℓ^∞ / c_0.)

It follows from this definition and Lemma II.25 that G is an Abelian semigroup on ℓ^∞ / c_0.

Now we apply Theorem II.23 to X and G as defined in the previous discussion, $p = \|\cdot\|_X$, $Y = \{[x] \in X : x = (a, a, a, ...)\}$ which is the equivalence classes of elements of c inside X, and $f : Y \to \mathbb{R}$ which maps $[x]$ to a where $x = (a, a, a, ...)$.

This gives an extension F of f with the property that $F([x]) \le \|[x]\|_{\ell^\infty / c_0}$ for all $[x] \in X$. Also, $F(T[x]) = F([x])$ for every $T \in G$ and every $[x] \in X$.

This induces a Banach limit $F_0 : \ell^\infty \to \mathbb{R}$ such that $F(Cx) = F(x)$ for every $x \in \ell^\infty$.

This construction is flexible. If we take some other summability method M and show that M commutes with C and S after modding out by c_0, then Theorem II.23 will give a Banach Limit respecting M and C and their compositions.

Given any sequence, $x = (x_1, x_2, ...)$, define $P(x)$ to be the sequence whose nth term is

$$P_n x = \sum_{k=1}^{n} \frac{1}{2^{n-1}} \binom{n-1}{k-1} x_k.$$

P is a regular summability method. We will add the Pascal operator P to the Banach limit above. We need one arithmetic lemma before proving that this is possible.

Lemma II.26. *For any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ with $k \leq n$, we have*

$$\sum_{j=k}^{n} \binom{n}{j} = 2^{n-1} \sum_{j=k}^{n} \frac{1}{2^{j-1}} \binom{j-1}{k-1}.$$

Proof. When $n = 1$, both left and right hand sides of the equation equal one. By way of induction, suppose that $n > 1$ and

$$\sum_{j=k}^{n-1} \binom{n-1}{j} = 2^{n-2} \sum_{j=k}^{n-1} \frac{1}{2^{j-1}} \binom{j-1}{k-1}.$$

Then, first we compute using Pascal's rule that when $j < n$ we have $\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$.

$$\begin{aligned}
\sum_{j=k}^{n} \binom{n}{j} &= 1 + \sum_{j=k}^{n-1} \binom{n}{j} = 1 + \sum_{j=k}^{n-1} \binom{n-1}{j-1} + \sum_{j=k}^{n-1} \binom{n-1}{j} \\
&= 1 + \sum_{m=k-1}^{n-2} \binom{n-1}{m} + \sum_{j=k}^{n-1} \binom{n-1}{j} \\
&= \binom{n-1}{n-1} + \sum_{m=k-1}^{n-2} \binom{n-1}{m} + \sum_{j=k}^{n-1} \binom{n-1}{j} \\
&= \sum_{m=k-1}^{n-1} \binom{n-1}{m} + \sum_{j=k}^{n-1} \binom{n-1}{j} = \binom{n-1}{k-1} + 2 \sum_{j=k}^{n-1} \binom{n-1}{j}.
\end{aligned}$$

Now we can apply the inductive hypothesis to this last quantity and get

$$\begin{aligned}\sum_{j=k}^{n}\binom{n}{j} &= \binom{n-1}{k-1} + 2\cdot 2^{n-2}\sum_{j=k}^{n-1}\frac{1}{2^{j-1}}\binom{j-1}{k-1} \\ &= 2^{n-1}\left(\frac{1}{2^{n-1}}\binom{n-1}{k-1} + \sum_{j=k}^{n-1}\frac{1}{2^{j-1}}\binom{j-1}{k-1}\right) \\ &= 2^{n-1}\sum_{j=k}^{n}\frac{1}{2^{j-1}}\binom{j-1}{k-1}.\end{aligned}$$

□

Now we need to show that P commutes with S and C in ℓ^∞/c_0. It turns out that P commutes with C without modding out by c_0.

Lemma II.27. *For every $(x_1, x_2, x_3, ...) \in \ell^\infty$, $PCx = CPx$.*

Proof. We are going to check this with a direct calculation. First note that the nth term of PCx is

$$(PCx)_n = \frac{1}{2^{n-1}}\sum_{j=1}^{n}\binom{n-1}{j-1}\frac{1}{j}\sum_{k=1}^{j}x_k.$$

and the nth term of CPx is

$$(CPx)_n = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{2^{j-1}}\sum_{k=1}^{j}\binom{j-1}{k-1}x_k.$$

Changing the order of summation we get

$$(PCx)_n = \frac{1}{2^{n-1}}\sum_{k=1}^{n}\left(\sum_{j=k}^{n}\binom{n-1}{j-1}\frac{1}{j}\right)x_k$$

and

$$(CPx)_n = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j=k}^{n}\frac{1}{2^{j-1}}\binom{j-1}{k-1}\right).$$

Setting $A_{n,k} = \frac{1}{2^{n-1}}\sum_{j=k}^{n}\binom{n-1}{j-1}\frac{1}{j}$ and $B_{n,k} = \frac{1}{n}\sum_{j=k}^{n}\frac{1}{2^{j-1}}\binom{j-1}{k-1}$ gives

$$(PCx)_n = \sum_{k=1}^{n}A_{n,k}x_k \quad \text{and} \quad (CPx)_n = \sum_{k=1}^{n}B_{n,k}x_k.$$

It turns out that $A_{n,k} = B_{n,k}$, which establishes the result. Indeed, applying Lemma II.26,

$$\begin{aligned}
A_{n,k} &= \frac{1}{n2^{n-1}} \sum_{j=k}^{n} \binom{n}{j} \\
&= \frac{1}{n2^{n-1}} 2^{n-1} \sum_{j=k}^{n} \frac{1}{2^{j-1}} \binom{j-1}{k-1} \\
&= \frac{1}{n} \sum_{j=k}^{n} \frac{1}{2^{j-1}} \binom{j-1}{k-1} = B_{n,k}.
\end{aligned}$$

□

Lemma II.28. *For every $x \in \ell^\infty$, $SPx - PSx \in c_0$.*

Proof. Recall that for any $x = (x_1, x_2, x_3, ...) \in \ell^\infty$ the image of this sequence under the Pascal operator is

$$Px = \left(x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{4}(x_1 + 2x_2 + x_3), \frac{1}{8}(x_1 + 3x_2 + 3x_3 + x_4), ... \right).$$

The nth term of this sequence is $P_n x = \sum_{k=1}^{n} \frac{1}{2^{n-1}} \binom{n-1}{k-1} x_k$. Because $Sx = (x_2, x_3, x_4, ...)$ it follows that $P_n Sx = \sum_{k=1}^{n} \frac{1}{2^{n-1}} \binom{n-1}{k-1} x_{k+1}$. Also, shifting Px gives that the nth term of SPx is $S_n Px = \sum_{k=1}^{n+1} \frac{1}{2^n} \binom{n}{k-1} x_k$ which (changing indices) is $S_n Px = \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} x_{k+1}$. Now we can check the nth term of $PSx - SPx$.

$$\begin{aligned}
P_n Sx - S_n Px &= \sum_{k=1}^{n} \frac{1}{2^{n-1}} \binom{n-1}{k-1} x_{k+1} - \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} x_{k+1} \\
&= -\frac{1}{2^n} \binom{n}{0} x_1 + \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}} \binom{n-1}{k-1} - \frac{1}{2^n} \binom{n}{k} \right) x_{k+1} \\
&= -\frac{1}{2^n} x_1 + \sum_{k=1}^{n} \left(\frac{1}{2^{n-1}} \frac{n!}{(k-1)!(n-k)!} - \frac{1}{2^n} \frac{n!}{k!(n-k)!} \right) x_{k+1} \\
&= -\frac{1}{2^n} x_1 + \sum_{k=1}^{n} \frac{1}{2^{n-1}} \frac{(n-1)!}{(k-1)!(n-k)!} \left(1 - \frac{1}{2} \frac{n}{k} \right).
\end{aligned}$$

The kth term of this last sum is negative if and only if $2k - n < 0$, or equivalently $k < n/2$. Note that $P_n Sx - S_n Px$ is of the form $\sum a_k x_k - \sum b_k x_k$ where $\sum a_k = 1$ and

$\sum b_k = 1$. So if we let $I_1 = \{k : a_k > b_k\}$ and $I_2 = \{k : a_k < b_k\}$ (note that these sets depend on n), then

$$|P_n Sx - S_n Px| \le \sum_{I_1 \cup I_2} |a_k - b_k| \|x\|_\infty = 2 \sum_{k \in I_2} (b_k - a_k) \|x\|_\infty.$$

We will finish the proof by showing that $\lim_{n\to\infty} \sum_{k\in I_2} (b_k - a_k) = 0$. First, suppose that $n = 2j$ is even. Then

$$\begin{aligned}
\sum_{k\in I_2} (b_k - a_k) &= \sum_{k=0}^{j-1} \frac{1}{2^{2j}} \binom{2j}{k} - \sum_{k=1}^{j-1} \frac{1}{2^{2j-1}} \binom{2j-1}{k-1} \\
&= \left(\frac{1}{2} - \frac{1}{2}\frac{1}{2^{2j}} \binom{2j}{j} \right) - \left(\frac{1}{2} - \frac{1}{2^{2j-1}} \binom{2j-1}{j-1} \right) \\
&= \frac{1}{2^{2j-1}} \binom{2j-1}{j-1} - \frac{1}{2}\frac{1}{2^{2j}} \binom{2j}{j} \le \frac{1}{2^{2j-1}} \frac{(2j-1)!}{(j-1)!j!}.
\end{aligned}$$

We can use Stirling's formula $\left(\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1 \right)$ to show that this last quantity converges to 0 as follows.

$$\begin{aligned}
0 \;&\le\; \lim_{j\to\infty} \frac{1}{2^{2j-1}} \frac{(2j-1)!}{(j-1)!j!} = \lim_{j\to\infty} \frac{1}{2^{2j-1}} \frac{\sqrt{2\pi(2j-1)}(\frac{2j-1}{e})^{2j-1}}{\sqrt{2\pi(j-1)}(\frac{j-1}{e})^{j-1}\sqrt{2\pi j}(\frac{j}{e})^j} \\
&= \lim_{j\to\infty} \frac{1}{2^{2j-1}} \frac{\sqrt{2j-1}}{\sqrt{j-1}\sqrt{2\pi j}} \frac{(2j-1)^{2j-1}}{(j-1)^{j-1} j^j} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{j\to\infty} \sqrt{\frac{2j-1}{(j-1)j}} \cdot \frac{(j-1/2)^{2j-1}}{(j-1)^{j-1} j^j} \\
&\le \lim_{j\to\infty} \sqrt{\frac{2}{j-1}} \left(\frac{j-1/2}{j-1} \right)^{j-1} \frac{j^j}{j^j} = \lim_{j\to\infty} \sqrt{\frac{2}{j-1}} \left(\frac{j-1+1/2}{j-1} \right)^{j-1} \\
&= \lim_{j\to\infty} \sqrt{\frac{2}{j-1}} \left(1 + \frac{1/2}{j-1} \right)^{j-1} = e^{1/2} \lim_{j\to\infty} \sqrt{\frac{2}{j-1}} = 0.
\end{aligned}$$

Now suppose that $n = 2j+1$ is odd. In this case

$$\begin{aligned}
\sum_{k\in I_2} (b_k - a_k) &= \sum_{k=0}^{j} \frac{1}{2^{2j+1}} \binom{2j+1}{k} - \sum_{k=1}^{j} \frac{1}{2^{2j}} \binom{2j}{k-1} \\
&= \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{2}\frac{1}{2^{2j}} \binom{2j}{j} \right) = \frac{1}{2^{2j+1}} \frac{(2j)!}{j!j!}.
\end{aligned}$$

It follows from Stirling's formula (we omit a calculation that is nearly identical to the one above) that this converges to 0 as $j \to \infty$. $\square$

Theorem II.29. *There is a Banach limit L so that for every $x \in \ell^\infty$ we have $L(Fx) = L(x)$ where F is any composition of any number of Cesàro and Pascal operators.*

Proof. It follows from the Lemmas showing that these operators commute up to c_0 and Theorem II.23. □

Open Question II.30. *What if we try to add G and G^{ω_0} to this mix?*

Open Question II.31. *Is it possible to incorporate other well-studied summability methods into Banach limits like the ones constructed above?*

G. CESÀRO AVERAGING AND THE MAZUR PRODUCT OF BOUNDED SEQUENCES

Given two scalar sequences x and y, we define their **Mazur Product** (see, for example, Mazur's Problem 8 in the Scottish Book [40]) to be the sequence $x \boxtimes y$ for which the n-th term is

$$(x \boxtimes y)_n := \frac{1}{n} \sum_{k=1}^{n} x_k \, y_{n-k+1} \ .$$

It is well-known that if x and y are convergent to λ and μ, respectively, then $x \boxtimes y$ converges to the product $\lambda \, \mu$. Consider the following interesting analogue of this result.

Theorem II.32. *If $x, y \in \ell^\infty$ both have convergent Cesàro averages, $Cx \to x^*$ and $Cy \to y^*$, then*

$$C(x \boxtimes y) \to x^* \, y^* \ .$$

We will prove this result, and more, below. We will use the notation $\mathbf{1}$ for the constant sequence $(1, 1, 1, ...)$. For any $a \in \mathbb{R}$, $a\mathbf{1}$ be the sequence $(a, a, a, ...)$.

Lemma II.33. *If $a \in \mathbb{R}$ and $z \in Ces_\lambda$ for some $\lambda \in \mathbb{R}$, then $a \cdot (Cz)_n = (z \boxtimes a\mathbf{1})_n$.*

This lemma follows from checking that

$$a \cdot (Cz)_n = a \frac{1}{n} \sum_{k=1}^{n} z_k = \frac{1}{n} \sum_{k=1}^{n} z_k \, a = (z \boxtimes a\mathbf{1})_n.$$

This lemma gives that when $z \in Ces_\lambda$, it follows that $(z \boxtimes a\mathbf{1})_n \to a \cdot \lambda$. This is the fact that we will use later.

The first major result of this type was proven in 1890 by Cesàro [13]. He showed that for convergent series $\sum a_n = A$ and $\sum b_n = B$, the Cauchy product $\sum c_n$ is Cesàro convergent to AB (here $c_n = \sum_{k=0}^n a_k b_{n-k}$). There is of course a very close relationship between the Cauchy and Mazur products. The Mazur product is a term-by-term average of the Cauchy product.

So one might expect to find versions of these lemmas in the literature. We include all details anyway, because our purpose is just to get the tools in place to discuss the product of our Banach limits constructed above.

Theorem II.34. *Suppose $x, y \in \ell^\infty$, with x convergent, $x \to x^*$ and $y \in Ces$, $(Cy)_n \to y^*$. Then $(x \boxtimes y)_n \to x^* y^*$*

Proof.

$$|(x \boxtimes y)_n - x^*y^*| = \left| \sum_{k=1}^n \frac{x_k y_{n+1-k}}{n} - x^*y^* \right|$$

$$\left| \sum_{k=1}^n \frac{x_k y_{n+1-k}}{n} - \sum_{k=1}^n \frac{x^* y_{n+1-k}}{n} + x^* \sum_{k=1}^n \frac{y_{n+1-k} - y^*}{n} \right|$$

$$\leq \sum_{k=1}^n \left| y_{n+1-k} \frac{x_k - x^*}{n} \right| + |x^*| \left| \sum_{j=1}^n \frac{y_j}{n} - y^* \right|$$

$$\leq \|y\|_\infty \sum_{k=1}^n \left| \frac{x_k - x^*}{n} \right| + |x^*| |(Cy)_n - y^*|.$$

The right term goes to zero because the partial averages of y converge. The left term goes to zero for the same reason that the Cesàro averages of a convergent sequence converge to the same limit. ∎

In fact this gives a class of extra sequences with $(x \boxtimes y)_n$ converging to some meaningful number.

Corollary II.35. *Suppose $x, y \in \ell^\infty$ with $x, y \in Ces$, $(Cx)_n \to x^*$, $(Cy)_n \to y^*$, and*

$$\sum_{k=1}^n \left| \frac{x_k - x^*}{n} \right| \underset{n}{\to} 0.$$

Then $(x \boxtimes y)_n \to x^* y^*$.

This corollary follows immediately from the proof above. Notice that we only need one of x or y to satisfy the new condition. The set of x satisfying this hypothesis is small in a sense, even as a subset of the sequences whose Cesàro averages converge. For example

$$x = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, ...)$$

works, but if we let

$$y = x = ((-1)^n)$$

then

$$(x \boxtimes y)_n = \frac{1}{n} \sum_{k=1}^{n} x_k y_{n-k+1} = \frac{1}{n} \sum_{k=1}^{n} (-1)^{n+1} = (-1)^{n+1}$$

which does not converge. Yet this sequence has a convergent average. This example was key in formulating the hypothesis of Theorem II.32. We need one more lemma before proving the theorem.

Lemma II.36. *Suppose* $x, y \in \ell^\infty$ *with* $(Cx) \to 0$*. Then* $C \lim(x \boxtimes y) = 0$*.*

Proof. Define $z := x \boxtimes y$ and note that $z \in \ell^\infty$.

$$z_n = \frac{1}{n} \sum_{k=1}^{n} x_k y_{n-k+1}.$$

$$\begin{aligned} (Cz)_m &= \frac{1}{m} \sum_{k=1}^{m} z_k = \frac{1}{m} \sum_{k=1}^{m} \frac{1}{n} \sum_{k=1}^{n} x_k y_{n-k+1} \\ &= \frac{1}{m} \sum_{k=1}^{m} x_k \sum_{n=k}^{m} \frac{y_{n-k+1}}{n} \\ &= \frac{1}{m} \sum_{k=1}^{m} x_k \sum_{j=1}^{m-k+1} \frac{y_j}{j+k-1}. \end{aligned}$$

If we define $u_k := \sum_{j=1}^{m-k+1} \frac{y_j}{j+k-1}$ then we can use Abel's formula to rewrite

$$\begin{aligned} (Cz)_m &= \frac{1}{m} \sum_{k=1}^{m} x_k \, u_k && \text{(II.1)} \\ &= \frac{1}{m} \left[\sum_{k=1}^{m-1} \left(\sum_{j=1}^{k} x_j (u_k - u_{k+1}) \right) + u_m \sum_{k=1}^{m} x_k \right]. && \text{(II.2)} \end{aligned}$$

Note that $(u_1 - u_2) = \frac{y_m}{m} + \sum_{j=1}^{m-1} \frac{y_j}{j(j+1)}$, $u_m = \frac{y_1}{m}$, and

$$u_k - u_{k+1} = \frac{y_{m-k+1}}{m} + \sum_{j=1}^{m-k} \frac{y_j}{(j+k-1)(j+k)}.$$

Let's look at the first term of (II.2).

$$\begin{aligned} \left|\frac{x_1(u_1-u_2)}{m}\right| &\leq \frac{|x_1|\|y\|_\infty}{m}\left(\frac{1}{m} + \sum_{j=1}^{\infty}\frac{1}{j(j+1)}\right) \\ &\leq \frac{2\|y\|_\infty}{m}(|(Cx)_1|). \end{aligned}$$

The last term of (II.2) is easier.

$$\left|\frac{1}{m}\frac{y_1}{m}\sum_{k=1}^{m} x_k\right| \leq \frac{2\|y\|_\infty}{m}\left|\sum_{k=1}^{m} x_k\right| = \frac{2\|y\|_\infty}{m}|(Cx)_m|.$$

Now, for $1 < k < m$ the kth term of (II.2) is

$$\begin{aligned} \left|\frac{u_k - u_{k+1}}{m}\sum_{j=1}^{k} x_j\right| &\leq \left|\frac{1}{m}\left(\frac{y_{m-k+1}}{m} + \sum_{j=1}^{m-k}\frac{y_j}{(j+k-1)(j+k)}\right)\sum_{j=1}^{k} x_j\right| \\ &\leq \frac{\|y\|_\infty}{m}\left(\frac{1}{m} + \sum_{j=1}^{\infty}\frac{1}{(j+k-1)(j+k)}\right)k|(Cx)_k| \\ &= \frac{\|y\|_\infty}{m}\left(\frac{1}{m} + \frac{1}{k}\right)k|(Cx)_k| \\ &\leq \frac{2\|y\|_\infty}{m}|(Cx)_k|. \end{aligned}$$

Putting these terms together gives

$$|(Cz)_m| \leq \frac{2\|y\|_\infty}{m}\sum_{k=1}^{m}|(Cx)_k|.$$

But we know that $|(Cx)_k| \underset{k}{\longrightarrow} 0$ meaning $\displaystyle\sum_{k=1}^{m}\frac{|(Cx)_k|}{m} \underset{m}{\longrightarrow} 0$ implying $(Cz)_m \underset{m}{\longrightarrow} 0$. $\square$

Now we can prove Theorem II.32.

Proof. Define the sequence $\tilde{x} = x - x^*\mathbf{1}$. Here $\mathbf{1} = (1, 1, 1, ...)$ and so $x^*\mathbf{1} = (x^*, x^*, x^*, ...)$. We can see that $\tilde{x} \in Ces_0$ by checking

$$(C\tilde{x})_n = \frac{1}{n}\sum_{k=1}^{n}(x_k - x^*) = \frac{1}{n}\sum_{k=1}^{n} x_k - \frac{1}{n}\sum_{k=1}^{n} x^* = (Cx)_n - x^* \underset{n}{\rightarrow} 0.$$

Now

$$\begin{aligned}
(C(x \boxtimes y))_n &= \frac{1}{n}\sum_{k=1}^{n}(x^* + (x_n - x^*))y_{n-k+1} \\
&= \frac{1}{n}\sum_{k=1}^{n} x^* y_{n-k+1} + \frac{1}{n}\sum_{k=1}^{n}(x_n - x^*)y_{n-k+1} \\
&= (C(x^*\mathbf{1} \boxtimes y))_n + (C(\tilde{x} \boxtimes y))_n.
\end{aligned}$$

The right hand term converges to zero using $\tilde{x} \in Ces_0$ in the previous theorem. Lemma II.33 gives that the left term is $x^* \cdot C(Cy)_n$, which converges to x^*y^*. $\square$

Corollary II.37. *Let φ be Banach limit which respects the Cesàro method as constructed above (i.e. $\varphi(Cz) = \varphi(z)$ for all $z \in \ell^\infty$). Let $x, y \in \ell^\infty$ be Cesàro convergent to x^* and y^* respectively. Then $\varphi(x \boxtimes y) = x^*y^*$.*

Proof. It follows immediately from the fact that $\varphi(x \boxtimes y) = \varphi(C(x \boxtimes y))$ and the fact that $C(x \boxtimes y) \to x^*y^*$. $\square$

Note that for any $x \in \ell^\infty$ we have that $x \boxtimes \mathbf{1} = Cx$. And so it follows that for any φ respecting the Cesàro method we have

$$\varphi(x \boxtimes \mathbf{1}) = \varphi(Cx) = \varphi(x) = \varphi(x)\varphi(\mathbf{1}).$$

This begs the following question.

Open Question II.38. *Is it the case that for every φ respecting the Cesàro method and every $x, y \in \ell^\infty$ we have $\varphi(x \boxtimes y) = \varphi(x)\varphi(y)$?*

Question II.39. *Do similar theorems (to II.32) hold for other regular summability methods and their associated Mazur-type products?*

For example, we can write down a product which generalizes the Pascal method.

For $x, y \in \ell^\infty$ (with x, y indexed by $\mathbb{N}_0$) define the **Pascal product** $z = x \maltese y$ by

$$z_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}.$$

This product is commutative. Also, letting $z = x \maltese \mathbf{1}$ and taking $\lim z_n$ is exactly Pascal's summability method applied to x.

Lemma II.40. *Suppose $x, y \in \ell^\infty$. If $x_n \to_n x_\infty$ and $P_n(y) \to_n y_\infty$, then $(x \maltese y)_n \to_n x_\infty y_\infty$.*

Proof.

$$(x \maltese y)_n - x_\infty y_\infty = \sum_{k=0}^{n} \binom{n}{k} \frac{x_k y_{n-k} - x_\infty y_\infty}{2^n}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{x_k y_{n-k} - x_\infty y_{n-k}}{2^n} + \sum_{k=0}^{n} \binom{n}{k} \frac{x_\infty y_{n-k} - x_\infty y_\infty}{2^n}$$

$$= A_n + B_n,$$

where A_n is the left sum, and B_n is the right sum.

We can see that

$$|A_n| \leq \|y\|_\infty \sum_{k=0}^{n} \binom{n}{k} \frac{|x_k - x_\infty|}{2^n}$$

$$= \|y\|_\infty P_n(\{|x_k - x_\infty|\}_{k \in \mathbb{N}_0}),$$

which converges to 0 by the regularity of the Pascal method.

Now for B_n. Note that when $x_\infty = 0$ we get $B_n = 0 \ \forall n$. Otherwise

$$\frac{B_n}{x_\infty} = \sum_{k=0}^{n} \binom{n}{k} \frac{y_{n-k}}{2^n} - y_\infty = P_n(y) - y_\infty.$$

This converges to zero by the assumption on $P_n(y)$. $\square$

Question II.41. *If $x, y \in \ell^\infty$ such that $P_n(x) \to x_\infty$ and $P_n(y) \to y_\infty$, does it follow that $P_n(x \maltese y) \to x_\infty y_\infty$?*

H. FUTURE WORK AND OPEN QUESTIONS

Question II.10: Can the bounds in Theorem II.9 be improved?

Question II.11: Are there versions of Theorem II.9 corresponding to other regular summability methods?

Question II.15: Which (non-Cesàro convergent) $x \in \ell^\infty$ are such that

$$\limsup C(C(x)) - \liminf C(C(x)) = \limsup C(x) - \liminf C(x)?$$

Question II.20: Is it the case that a Banach limit recognizing the convergence of every iterate of G will also recognize G^{ω_0} convergence?

Question II.21: Considering the previous question, is it possible to extend the construction to higher ordinals?

Questions II.30 and II.31: Can G and G^{ω_0} be added to the Banach limits that respect P and C? Is it possible to incorporate other well-studied summability methods?

Question II.38: Is it the case that for every φ respecting the Cesàro method and every $x, y \in \ell^\infty$ we have $\varphi(x \boxtimes y) = \varphi(x)\varphi(y)$?

Question II.39: Do theorems similar to II.32 hold for other regular summability methods and their associated Mazur-type products?

Question: We saw in Chapter I that iterates (even up through the ordinal numbers) can lead to answers to the fixed point question. Is there a situation where applying a summability method, its iterates, or a generated Banach limit to a sequence of iterates of a nonexpansive map or an approximate fixed point sequence will lead to a fixed point?

Question II.41: If $x, y \in \ell^\infty$ such that $P_n(x) \to x_\infty$ and $P_n(y) \to y_\infty$, does it follow that $P_n(x \maltese y) \to x_\infty y_\infty$?

Question: Can some of the work of this chapter be extended to include sequence spaces that include some unbounded sequences?

III. DIFFERENTIABILITY

A. DEFINITIONS AND SUMMARY OF RESULTS

We will use the standard definition of Fréchet derivative for a function between Banach spaces, which is as follows.

Definition III.1. *Given Banach spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *and a function* $f : X \to Y$*, we say that* f *is* ***Fréchet differentiable*** *at* $c \in X$ *if there exists a* $T = T_c \in L(X, Y)$ *such that*

$$\lim_{h \to 0} \frac{\|f(c+h) - f(c) - T_c(h)\|_Y}{\|h\|_X} = 0.$$

In the 19th Century ([44], [45]), Giuseppe Peano introduced a definition for the derivative of a real valued function of a real variable which is formally stronger than the usual definition. This definition has been generalized in multiple directions. The basic results have been generalized to infinitesimals ([38]) and to Banach Spaces ([34] and [43]). We will call this the *strong derivative* (or *Peano derivative*) and define it as follows. We use the same definition as [34], [35], [28], and [43]. Bourbaki ([6]), Zajíček ([51]), and others use the term *strict derivative*.

Definition III.2. *Given Banach spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *and a function* $f : X \to Y$*, we say that* f *is* ***strongly differentiable*** *at* $c \in X$ *if there is a* $T = T_c \in L(X, Y)$ *such that*

$$\lim_{h \neq k,\, h,k \to 0} \frac{\|f(c+h) - f(c+k) - T_c(h-k)\|_Y}{\|h-k\|_X} = 0.$$

This definition is indeed stronger than the usual one. For example, $f(x) = x^2 \sin(1/x)$ can be extended to be differentiable at 0, but not strongly differentiable. In fact, Peano's strong derivative (in $\mathbb{R}$ and in all other Banach spaces) is continuous wherever it exists.

Conversely, whenever the Fréchet derivative is continuous on an open set, it follows that the strong derivative exists and equals the Fréchet derivative. This is true in all Banach spaces. For details of this and some other results see [45], [43], and [34].

In the case that $f : \mathbb{R} \to \mathbb{R}$, notice that the difference quotient

$$\frac{f(c+h) - f(c+k)}{h-k}$$

corresponding to f at c and varying with respect to h and k admits the slopes of a greater collection of secant lines than the usual definition. This gives the heuristic argument that the existence of the Peano derivative is stronger than the existence of the usual derivative.

Another reformulation of the derivative is used by Carathéodory [9] and others. This formula pivots on the mean value theorem, and was generalized to Hilbert spaces in [1]. Our main result in this chapter is to further generalize this definition to Banach spaces, and to combine the strong and Carathéodory derivatives in Banach spaces. The generalization permits some simpler proofs of facts and offers an alternate perspective on derivatives in Banach spaces.

Definition III.3. *Given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, an open set $U \subseteq X$, and a function $f : U \to Y$, we say that f is* ***Carathéodory differentiable*** *at $c \in U$ if there is a* ***difference transform*** *$\Gamma : U \to L(X, Y)$ that is continuous at c such that for all $x \in U$*

$$f(x) - f(c) = \Gamma(x)(x - c).$$

Definition III.4. *Given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, an open set $U \subseteq X$ and a function $f : U \to Y$, we say that f is* ***strongly Carathéodory differentiable*** *at $c \in U$ if there is a* ***strong difference transform*** *$\Gamma : U \times U \to L(X, Y)$ that is continuous at (c, c) such that $\forall x, y \in U$*

$$f(x) - f(y) = \Gamma(x, y)(x - y).$$

Now that we have two notions of strong derivatives, we will call the first (Definition III.2) the **Strong Fréchet Derivative**. We will show that in certain settings the derivatives (resp. strong derivatives) are equivalent. In these settings we can just say "differentiable" (resp. "strongly differentiable") without confusion.

B. MAIN RESULTS

Theorem III.5. *Let X and Y be Banach spaces. Let $c \in X$. Let $f : X \to Y$ be given. The following are equivalent:*

(1) f is Carathéodory differentiable at c.

(2) f is Fréchet differentiable at c.

Proof. [**(1)⇒(2):**] Assume that such a Γ exists. We claim that $\Gamma(c)$ is the Fréchet derivative of f at c. Indeed, consider the difference quotient

$$\frac{\|f(x) - f(c) - \Gamma(c)(x-c)\|_Y}{\|x-c\|_X} = \frac{\|(\Gamma(x) - \Gamma(c))(x-c)\|_Y}{\|x-c\|_X} \leq \|\Gamma(x) - \Gamma(c)\|_{L(X,Y)}.$$

Because Γ is continuous at c, we have that this difference quotient converges to 0 as $x \to c$. Thus f is Fréchet differentiable at c with Fréchet derivative $\Gamma(c)$.

[**(2)⇒(1):**] Now assume that f is Fréchet differentiable at c. That is, there exists a $T_c \in L(X,Y)$ such that

$$\lim_{x \to c} \frac{\|f(x) - f(c) - T_c(x-c)\|_Y}{\|x-c\|_X} = 0.$$

Define $\Gamma(c) = T_c$. For each $x \in X$ with $x \neq c$, let φ_x be any element of the unit sphere of X^* such that $\varphi_x(x-c) = \|x-c\|$. Then define

$$(\Gamma(x))(z) = \frac{(f(x) - f(c) - T_c(x-c))\varphi_x(z)}{\|x-c\|} + T_c(z).$$

Now that Γ is defined, it remains to be shown that this Γ satisfies the definition of difference transform for f at c. First, for each $x \in X$, recalling that $\varphi_x(x-c) = \|x-c\|$

$$\Gamma(x)(x-c) = \frac{(f(x) - f(c) - T_c(x-c))\varphi_x(x-c)}{\|x-c\|} + T_c(x-c) = f(x) - f(c).$$

Next, let us confirm the continuity of Γ at c. Recall that $\|\varphi_x\|_{X^*} = 1$. Then

$$\begin{aligned} \|\Gamma(x) - \Gamma(c)\|_{L(X,Y)} &= \|\Gamma(x) - T_c\|_{L(X,Y)} = \left\| \frac{f(x) - f(c) - T_c(x-c)}{\|x-c\|_X} \varphi_x \right\|_{L(X,Y)} \\ &= \frac{\|f(x) - f(c) - T_c(x-c)\|_Y}{\|x-c\|_X} \|\varphi_x\|_{X^*} \to 0. \end{aligned}$$

Finally, to see that $\Gamma(x) \in L(X,Y)$ note that $T_c \in L(X,Y)$ by definition and the first term is the product of a fixed vector in Y with a continuous linear functional on X. $\square$

One can see from the construction of the difference transform Γ that it may be highly non-unique. In higher dimensional Banach spaces, the set of candidates for φ_x is large. The only restriction is that $\Gamma(c) = T_c$. This fact is formalized in the following lemma.

Lemma III.6. *If Φ is any difference transform for f at c, then $\Phi(c) = T_c$.*

Proof. Let Φ and Ψ be any two difference transforms for f at c, and let $\eta(x) = \Phi(x) - \Psi(x)$. First note that $\forall x \in X$, $\eta(x)(x-c) = f(x) - f(c) - (f(x) - f(c)) = 0$. Then $\forall x$,

$$\|\eta(c)(x-c)\| = \|(\eta(c) - \eta(x))(x-c)\| \leq \|\eta(c) - \eta(x)\|_{op} \|x-c\|_X$$

$$\Rightarrow \left\| \eta(c) \frac{x-c}{\|x-c\|} \right\| \leq \|\eta(c) - \eta(x)\|_{op}.$$

Now let $u \in X$ be any unit vector. Putting $x = c + t\,u$ into the above inequality, and recalling that η is continuous at c gives

$$\|\eta(c)(u)\| \leq \|\eta(c) - \eta(c + t\,u)\|_{op} \to_{t \to 0} 0.$$

Thus $\eta(c)(u) = 0$ for any unit vector u, meaning $\Phi(c) = \Psi(c)$. In particular, if $f : X \to Y$ is differentiable at $c \in X$ in the sense of Carathéodory, then $\Gamma(c) = Df(c)$. □

Taking a cue from [1], one can now develop the theory of differentiation in Banach spaces more simply. In many proofs, the simplification comes from the fact that the analysis of limits is in the proofs of the equivalence of the Carathéodory and Fréchet derivatives and the previous lemma. A proof of the Banach space chain rule is included to demonstrate the point. See [42], [10], or [47] for comparison.

Corollary III.7. *If $g : X \to Y$ and $f : Y \to Z$ so that g is Fréchet differentiable at $c \in X$ and f is Fréchet differentiable at $g(c) \in Y$, then*

$$(Df(g))(c) = Df(g(c)) \circ Dg(c).$$

Proof. We know that Fréchet differentiable implies Carathéodory. Now, if ϕ and ψ are difference transforms for f at $g(c)$ and g at c respectively, then

$$f(g(x)) - f(g(c)) = \phi(g(x))(g(x) - g(c)) = \phi(g(x))\psi(x)(x - c).$$

ϕ is continuous at $g(c)$ while ψ and g are continuous at c, so $\phi(g(\cdot))\psi(\cdot)$ is continuous at c. This establishes that the composition is Carathéodory differentiable. Therefore, Theorem III.5 tells us that it is also Fréchet differentiable, and Lemma III.6 tells us that the derivative is of the expected form. $\square$

As previously mentioned, one well known result is the equivalence of strong Fréchet derivatives and continuous derivatives. We extend this result to strong Carathéodory derivatives by showing that the two notions of strong derivatives are the same on open sets.

Theorem III.8. *Let X and Y be Banach spaces. Suppose $f : X \to Y$ and $U \subseteq X$ is open. The following are equivalent:*

(0) f is Fréchet differentiable on U, and the derivative is continuous on U,

(1) f is strongly Fréchet differentiable on U, and

(2) f is strongly Carathéodory differentiable on U.

Proof. [**(0)⇔(1):**] This is proven in [43].

[**(2)⇒(1):**] Fix $c \in U$. Assume the corresponding difference transform, Γ, exists. We claim that $\Gamma(c, c)$ is the strong Fréchet derivative of f at c.

$\forall x, y \in U$ with $x \neq y$

$$\begin{aligned}\frac{\|f(x) - f(y) - \Gamma(c, c)(x - y)\|_Y}{\|x - y\|_X} &= \frac{\|(\Gamma(x, y) - \Gamma(c, c))(x - y)\|_Y}{\|x - y\|_X} \\ &\leq \|\Gamma(x, y) - \Gamma(c, c)\|_{L(X,Y)}.\end{aligned}$$

By our assumption that Γ is continuous at the diagonal we have that this last quantity approaches 0 as $(x, y) \to (c, c)$. This gives the strong Fréchet differentiability of f at (c, c).

[**(1)⇒(2):**] Assume f is strongly Fréchet differentiable at c for every $c \in U$. For every c, let $T_c \in L(X, Y)$ be such that

$$\lim_{(x,y)\to(c,c),x\neq y} \frac{\|f(x)-f(y)-T_c(x-y)\|_Y}{\|x-y\|_X} = 0.$$

Now we can define a Γ that recognizes strong Carathéodory differentiability at c. Define $\Gamma(x,x) = T_x$ for any $x \in U$. If $x \neq y$, let $\varphi_{(x,y)}$ be any element of the unit sphere of X^* such that $\varphi_{(x,y)}(x-y) = \|x-y\|_X$. Then if $x \neq y$ we can define

$$\Gamma(x,y)(z) := T_y(z) + \frac{(f(x)-f(y)-T_y(x-y))\varphi_{x,y}(z)}{\|x-y\|_X}.$$

As before, $\Gamma(x,y)$ is a continuous linear function for all x, y, and $\Gamma(x,y)(x-y) = f(x) - f(y)$.

First, consider the diagonal. As $(x,x) \to (c,c)$ we have $\Gamma(x,x) = T_x \to T_c = \Gamma(c,c)$.

Secondly, consider $(x,y) \in U \times U$ with $x \neq y$. Then

$$\begin{aligned}
\|\Gamma(x,y)-\Gamma(c,c)\|_{L(X,Y)} &= \left\| T_y(\cdot) + \frac{f(x)-f(y)-T_y(x-y)}{\|x-y\|_X}\varphi_{x,y}(\cdot) - T_c(\cdot)\right\|_{L(X,Y)} \\
&\leq \|T_y - T_c\|_{L(X,Y)} + \frac{\|f(x)-f(y)-T_y(x-y)\|_Y}{\|x-y\|_X}\|\varphi_{x,y}\|_{X^*} \\
&= \|T_y - T_c\|_{L(X,Y)} + \frac{\|f(x)-f(y)-T_y(x-y)\|_Y}{\|x-y\|_X}.
\end{aligned}$$

Now let $(x,y) \to (c,c)$ off of the diagonal. Here the left term approaches 0 by the continuity of the Fréchet derivative and the right term approaches zero by the definition of strong derivative.

Therefore Γ is continuous at (c,c). $\square$

C. STRONG DERIVATIVE AND CONTINUOUS GÂTEAUX DERIVATIVE

Definition III.9. *Let $f : X \to Y$. We say that f is **Gâteaux differentiable** at a point $x \in X$ if there is a bounded linear function $\delta f(x,\cdot) : X \to Y$ such that for every **direction** $h \in X$ we have*

$$\lim_{t\to 0}\left\|\frac{f(x+th)-f(x)}{t} - \delta f(x,h)\right\|_Y = 0.$$

It is known that all Fréchet differentiable functions are Gâteaux differentiable. It is also known that not all Gâteaux differentiable functions are Fréchet differentiable. However, we have the following well known theorem (see [49, p.21]).

Theorem III.10. *If the Gâteaux derivative of $f : X \to Y$ exists in a neighborhood of x_0, and δf is continuous at x_0 in operator norm, i.e. in the sense that*

$$\lim_{\epsilon \to 0} \|\delta f(x_0, \cdot) - \delta f(x_0 + \epsilon, \cdot)\|_{op} = 0,$$

then the Fréchet derivative of f exists at x_0 and $(Df(x_0))(h) = \delta f(x_0, h)$.

Proof. Let x_0 be the point given in the hypotheses, and suppose $B_r(x_0)$ is contained in the neighborhood of continuity of δf. Then $\forall h$ with $\|h\| < r$ the mean value theorem says that there is a $t_h \in (0, 1)$ such that

$$f(x_0 + h) - f(x_0) = \delta f(x_0 + t_h h, h).$$

To show that f is Fréchet differentiable at x_0 with derivative $\delta f(x_0, h)$, we compute (for $\|h\| < r$)

$$\frac{\|f(x_0 + h) - f(x_0) - \delta f(x_0, h)\|_Y}{\|h\|_X} = \frac{\|\delta f(x_0 + t_h h, h) - \delta f(x_0, h)\|_Y}{\|h\|_X}$$

$$\leq \frac{\|\delta f(x_0 + t_h h, \cdot) - \delta f(x_0, \cdot)\|_{op} \|h\|_X}{\|h\|_X} = \|\delta f(x_0 + t_h h, \cdot) - \delta f(x_0, \cdot)\|_{op} \to_h 0.$$

$\square$

Theorem III.11. *Given the hypotheses of the previous Theorem, the function is strongly Fréchet differentiable at x_0.*

Proof. Again, let $r > 0$ be such that $B_r(x_0)$ is contained in the domain of continuity of x_0. Then, whenever h and k are in X with norm less that r we apply the mean value theorem to find $t = t_{h,k} \in (0,1)$ such that

$$f(x_0 + h) - f(x_0 + k) = \delta f(x_0 + th + (1-t)k, h - k).$$

As before, we compute

$$\begin{aligned}
&\frac{\|f(x_0 + h) - f(x_0 + k) - \delta f(x_0, h-k)\|_Y}{\|h-k\|_X} \\
&= \frac{\|\delta f(x_0 + th + (1-t)k, h-k) - \delta f(x_0, h-k)\|_Y}{\|h-k\|_X} \\
&\leq \|\delta f(x_0 + th + (1-t)k, \cdot) - \delta f(x_0, \cdot)\|_{op} \to_{h,k} 0.
\end{aligned}$$

□

Theorem III.12. *Let $f : U \to Y$ be Gâteaux differentiable at every $x \in U$, and let $c \in U$. If f is strongly Fréchet differentiable at c, then $\forall h \in X$, $D_h f : U \to Y$ is continuous at c.*

Proof. Fix an arbitrary sequence $\{q_n\} \subseteq U$ such that $q_n \to c$, i.e. $\|q_n - c\|_X \to 0$. Let $h \in X$ with $h \neq 0$ be given. We want to show that $D_h f(q_n) \to D_h f(c)$. Recall that for any $n \in \mathbb{N}$

$$D_h f(q_n) = \lim_{t \to 0} \frac{f(q_n + th) - f(q_n)}{t}.$$

For $\epsilon = 1/n$ there exists $\delta_n \in (0, 1/n)$ such that $\forall t \in \mathbb{R}$ with $t \neq 0$ and $|t| < \delta_n$ we have $q_n + t/, h \in U$ and

$$\Gamma_n(t) := \left| D_h f(q_n) - \frac{f(q_n + t\,h)}{t} \right| < \frac{\|h\|_X}{n}.$$

Choose some $t_n \in \mathbb{R}$ with $t_n \neq 0$ such that $|t_n| < \delta_n$. Then $\forall n \in \mathbb{N}$

$$\begin{aligned}
D_h f(q_n) &= D_h f(q_n) - \frac{f(q_n + t_n\,h) - f(q_n)}{t_n} - \frac{f(q_n + t_n\,h) - f(q_n)}{t_n} \\
&= \Gamma_n(t_n) + \frac{f(q_n + t_n\,h) - f(q_n)}{t_n}.
\end{aligned}$$

Note that $|\Gamma_n(t_n)| < \|h\|_X/n \Rightarrow \Gamma_n(t_n) \to 0$. Also note that we can write

$$\frac{f(q_n + t_n\,h) - f(q_n)}{t_n} = \frac{f(c + (q_n - c + t_n\,h)) - f(c + (q_n - c))}{t_n}.$$

Recall that $t_n \neq 0$, $t_n \to 0$, and $q_n \to c$. This means $k_n := q_n - c \to 0$, $j_n := q_n - c + t_n\, h$, and $j_n - k_n = t_n\, h \neq 0$. Then because f is strongly differentiable at c there is $T_c \in L(X,Y)$ such that

$$Q_n := \frac{\|f(c+j_n) - f(c+k_n) - T_c(t_n\, h)\|_Y}{t_n\|h\|_X} \to 0.$$

Now we can write

$$D_h f(q_n) - D_h f(c) = \Gamma_n(t_n) + \frac{f(c+j_n) - f(c+k_n)}{t_n} - D_h f(c)$$

where

$$\left\| \frac{f(c+j_n) - f(c+k_n)}{t_n} - D_h f(c) \right\|_Y = \left\| \frac{f(c+j_n) - f(c+k_n)}{t_n} - T_c(h) \right\|_Y$$

$$= \frac{\|f(c+j_n) - f(c+k_n) - T_c(t_n\, h)\|_Y}{|t_n|\|h\|_X}\|h\|_X = Q_n\|h\|_X \to 0.$$

These inequalities combine to show

$$\|D_h f(c) - D_h f(q_n)\|_Y \leq \|h\|_X(1/n + Q_n) \to 0 \Rightarrow D_h f(q_n) \to D_h f(c).$$

□

D. CARATHÉODORY AND THE SELECTION OF SUPPORT FUNCTIONALS

The proof of Theorem III.8 hinges on the selection of a *support functional* φ for $(x-y)/\|x-y\|$ whose existence is guaranteed by the Hahn-Banach Theorem. Here we use the definition of support functional found in [17].

Definition III.13. *Given a Banach space X and a point $x \in S(X)$, we say that $\varphi_x \in S(X^*)$ is a support functional for x if $\varphi_x(x) = 1$.*

If we have a strongly Fréchet differentiable f as in Theorem III.8, and we can select this support functional continuously with respect to x and y, then there will be a difference transform Γ which is continuous on all of $U \times U$. The continuity of Γ can be seen by simply plugging this continually selected $\varphi_{x,y}$ into the definition of Γ in the proof of Theorem III.8.

The question of when one can select these support functionals continuously has been addressed in the literature.

Lemma III.14. *If X admits an equivalent Fréchet differentiable norm, then the support mapping $x \to \varphi_x$ is norm-to-norm continuous from $S(X)$ to $S(X^*)$ and unique.*

This Lemma is Theorem 1 of [17, p.30] (noting that property (iii) therein passes back to the original norm), with the guarantee of uniqueness coming from the weaker condition of Gâteaux differentiability. Note that the class of spaces admitting a Fréchet differentiable norm is quite large. Ekeland and Lebourg [23] proved that every such space is an Asplund space. Corollary 3.3 of [16] says that if X^* is separable, then X admits an equivalent Fréchet differentiable norm.

Now we can reformulate Theorem III.8. Note that once we have a difference transform that is continuous off the diagonal our result will be much closer to the Hilbert space result in [1].

Theorem III.15. *Let X and Y be Banach spaces, with X admitting an equivalent Fréchet differentiable norm. Suppose $f : X \to Y$ and $U \subseteq X$ is open. The following are equivalent:*

(1) f is strongly Fréchet differentiable on U.

(2) $\exists \Gamma : U \times U \to L(X, Y)$ such that Γ is continuous on $U \times U$ and

$$\Gamma(x, y)(x - y) = f(x) - f(y).$$

Proof. **[(2)⇒ (1):]** This condition (2) is stronger than the condition of the same number from Theorem III.8, which was seen there to imply (1).

[(1)⇒ (2):] For each point $c \in U$, let T_c be the (strong) Fréchet derivative for f at c. As before, define $\Gamma(c, c) = T_c$.

If $x \neq y$, then we want to pick a support functional for $(x-y)/\|x-y\|$, just as in Theorem III.8. By Lemma III.14 the only selection is continuous with respect to x and y. Define $\varphi_{x,y}$ to be the support functional for $(x-y)/\|x-y\|$. Then define

$$\Gamma(x,y)(z) := T_y(z) + \frac{(f(x) - f(y) - T_y(x-y))\varphi_{x,y}(z)}{\|x-y\|_X}.$$

In the proof of Theorem III.8 we saw that Γ satisfies the algebraic condition, and is continuous at the diagonal. It remains to see that this Γ is continuous off the diagonal.

Suppose $a \neq b$. Then $\exists$ a neighborhood of (a, b) in $U \times U$, which misses the diagonal. On this neighborhood $\Gamma(x, y)$ is defined above as a composition of functions (now including φ) which vary continuously with respect to x and y. Thus Γ varies continuously ($X \times X$-to-operator) with respect to x and y. $\square$

Open Question III.16. *Can we have this off-diagonal continuity in a space that does not admit an equivalent Fréchet differentiable norm?*

E. CARTAN AND STRONG DERIVATIVES

The phrase *strong derivative* appears in many places. Sometimes it is used to refer to the regular Fréchet derivative when in a setting with distributional derivatives. As mentioned above, we trace our defintion of *strong derivative* to Peano in [45]. Another definition of strong derivative is given in [10] by Henri Cartan. Cartan defines strongly differentiable at a point in the following way.

Definition III.17. *A function $f : X \to Y$ is **strongly Cartan differentiable** at a point $a \in X$ if there is a function $T \in L(X, Y)$ such that for any $\epsilon > 0$ there is an $r > 0$ such that whenever $\|x - a\| < r$ it follows that*

$$\|f(x) - f(a) - T(x-a)\| \leq \epsilon\|x - a\|.$$

One of the principle uses of the strong derivative from earlier was to localize the continuity condition in the inverse function theorem. The proof of the inverse function theorem is also Cartan's purpose for defining a strong derivative. His primary definition of differentiability follows.

Definition III.18. *A function $f : U \to Y$ is **Cartan differentiable** at the point $a \in U$ if the following conditions are satisfied:*

(i) f is continuous at the point a;

(ii) there exists a (continuous) linear map $T : X \to Y$ such that

$$\lim_{r>0, r\to 0} \frac{\sup_{\|x-a\|<r} \|f(x) - f(a) - T(x-a)\|}{r} = 0.$$

Note that Cartan does not explicitly require the linear map T to be continuous, but he remarks that it follows from the continuity of f. Also, Cartan notes (page 20, equation 2.1.3, [10]) that this last definition is the same as our definition of regular Fréchet derivative. It turns out, as proven below, that his definitions of derivative and strong derivative are also the same. Among other things, this makes Theorem 3.8.1 on page 49 of [10] redundant.

Theorem III.19. *The definitions of Cartan differentiable and strongly Cartan differentiable given above are equivalent.*

Proof. Suppose that f and a are such that Definition III.18 is satisfied, i.e. that f is differentiable at a in Cartan's usual sense. Suppose that f simultaneously fails to be strongly Cartan differentiable at a. Let $T \in L(X, Y)$ be the function recognizing the differentiability of f at a.

Because f is not strongly Cartan differentiable $\exists \epsilon > 0$ such that $\forall r > 0$ $\exists x$ with $\|x-a\| < r$ such that

$$\|f(x) - f(a) - T(x-a)\| > \|x-a\|\epsilon.$$

For each $n \in \mathbb{N}$ let x_n be such that $\|x_n - a\| < 1/n$ and $\|f(x_n) - f(a) - T(x_n - a)\| > \|x_n - a\|\epsilon$. It follows, for each n, that $x_n \neq a$ so $\exists$ maximal $m_n \in \mathbb{N}$ with the property that $\|x_n - a\| < 1/m_n$. Define $r_n = 1/m_n$ and note that $1/m_n \leq 1/n$ for each n. This implies $r_n \to_n 0$.

Now for any n we have

$$\sup_{\|x-a\|<r_n} \|f(x) - f(a) - T(x-a)\| \geq \|f(x_n) - f(a) - T(x_n - a)\| > \|x_n - a\|\epsilon \geq$$

$$\geq \frac{1}{m_n+1}\epsilon = \frac{m_n}{m_n+1}\frac{1}{m_n}\epsilon = \frac{m_n}{m_n+1}r_n\epsilon.$$

This implies that for any n

$$\frac{\sup_{\|x-a\|<r_n} \|f(x) - f(a) - T(x-a)\|}{r_n} > \frac{1}{2}\epsilon.$$

Again, $r_n \to 0$ so the limit in Defintion III.18 cannot be zero, a contradiction. So we see it is impossible for a function to be differentiable at a point and not strongly Cartan differentiable at that point. This is all to say that Definition III.18 implies Definition III.17.

Now for the converse. Here we should expect the argument to be easier.

Indeed, suppose that f, a, and T satisfy Definition III.17. Then for any $\epsilon > 0$, $\exists r_0 > 0$ such that whenever $\|x - a\| < r_0$ it follows that

$$\|f(x) - f(a) - T(x-a)\| \leq \epsilon\|x - a\| < \epsilon r_0.$$

We can extend this to say that for any $r < r_0$ and $\forall \|x - a\| < r < r_0$

$$\|f(x) - f(a) - T(x-a)\| \leq \epsilon\|x - a\| < \epsilon r,$$

meaning

$$\frac{\|f(x) - f(a) - T(x-a)\|}{r} < \epsilon.$$

This implies that $\forall 0 < r < r_0$

$$\frac{\sup_{\|x-a\|<r} \|f(x) - f(a) - T(x-a)\|}{r} \leq \epsilon.$$

Finally, $\epsilon > 0$ being arbitrary tells us that the limit in Definition III.17 is 0. $\square$

There is another result in Cartan's book (3.6.1 on page 44 of [10]) which is of interest to us. This result uses a mean value theorem to prove a fact about the convergence of the derivatives of a sequence of functions. By making similar assumptions about the convergence of difference transforms rather than derivates, we can make weaker assumptions about the domain and also get stronger conclusions about the convergence of the corresponding functions.

Theorem III.20. *Let X and Y be Banach spaces and let $U \subseteq X$ be open and convex. Let $f_n : U \to Y$ be a sequence of differentiable maps. Suppose there exists a point $a \in U$ such that the sequence $\{f_n(a)\} \subset Y$ has a limit $(f(a)) \in Y$. Finally, suppose that the sequence of mappings $Df_n : U \to L(X,Y)$ converges uniformly in U to a function $g : U \to L(X,Y)$.*

Then for any $x \in U$ the sequence $\{f_n(x)\}$ has a limit $(f(x))$, this convergence is uniform on any bounded part of U, and the limit function f is differentiable, and its derivative Df equals g.

In two new results we use the Carathéodory derivative in Banach spaces in order to re-frame this fact. Notice that we are able to drop the assumption of convexity. Also, the next result is not possible (even on a convex domain) if we restrict ourselves to Fréchet derivatives.

Theorem III.21. *Let X and Y be Banach spaces and let $U \subseteq X$ be open with $a \in U$ given. Let $f_n : U \to Y$ be a sequence of maps which are Carathéodory differentiable at a. Suppose that the sequence $\{f_n(a)\} \subset Y$ has a limit $(f(a)) \in Y$ and the difference transforms $\Gamma_n : U \to L(X,Y)$ corresponding to a are uniformly Cauchy. Then for any $x \in U$ the sequence $\{f_n(x)\}$ has a limit $(f(x))$.*

Proof. Let $a \in U$ be the point given in the assumptions and fix any $x \in U$. Then

$$\|f_n(x) - f_m(x)\|_Y = \|f_n(x) - f_n(a) + f_n(a) - f_m(a) + f_m(a) - f_m(x)\|$$

$$\leq \|f_n(a) - f_m(a)\| + \|f_n(x) - f_n(a) + f_m(a) - f_m(x)\|.$$

The first terms tends to 0 independently of x. The second term is

$$\|f_n(x) - f_n(a) - (f_m(x) - f_m(a))\|_Y$$

$$= \|\Gamma_n(x)(x-a) - \Gamma_m(x)(x-a)\|_Y \leq \|\Gamma_n(x) - \Gamma_m(x)\|_{op}\|x-a\|_X$$

$$\leq \|\Gamma_n(x) - \Gamma_m(x)\|_\infty \|x-a\|_X \to 0.$$

□

Theorem III.22. *Suppose that all assumptions of Theorem III.21 hold, and in addition that every f_n is Carathéodory differentiable at every $c \in U$ and the corresponding sequence of difference transforms $\Gamma_n : U \to L(X,Y)$ converges uniformly in U to a function $g_c : U \to L(X,Y)$.*

Then it follows that limit function f, guaranteed by Theorem III.21 is Carathéodory differentiable at every point and g_c is a difference transform for f at any given point c. Further, the convergence of the difference transforms corresponding to any bounded part of U is uniform.

Proof. Fix any $c \in U$. For any $x \in U$

$$\begin{aligned} f(x) - f(c) &= f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c) \\ &= (f_n(x) - f_n(c)) + (f(x) - f_n(x) + f_n(c) - f(c)). \end{aligned}$$

The right hand term tends to 0 in a way that depends on x and c. The left hand term is

$$\begin{aligned} f_n(x) - f_n(c) &= \Gamma_n(x)(x-c) \\ &\to g_c(x)(x-c), \end{aligned}$$

where Γ_n above corresponds to c. This all says that

$$f(x) - f(c) = g_c(x)(x-c)$$

and we know that g_c is continuous at c because it is the uniform limit of functions which are continuous at c

So f is Carathéodory differentiable at every point c in U.

Finally, let W be a bounded part of U, say $W \subseteq B_R(a)$. Then $\forall x \in W$ we have

$$\|f_n(x) - f(x)\|_Y = \|f_n(x) - f_n(a) + f_n(a) - f(a) + f(a) - f(x)\|$$

$$\leq \|f_n(x) - f_n(a) - (f(x) - f(a))\| + \|f_n(a) - f(a)\|.$$

The second term converges to 0 independently of x. Considering the Γ_n corresponding to a the 1st term is

$$\|\Gamma_n(x)(x-a) - g_a(x)(x-a)\| \leq \|\Gamma_n - g_a\|_\infty \|x-a\|_X$$

$$\leq R\|\Gamma_n - g_a\|_\infty \to 0$$

with this convergence depending only on x_0. So we see that the convergence is uniform on bounded parts of U. $\square$

To complete this illustration of the difference between the Fréchet and Carathéodory derivative, consider the following example.

Example III.23. *Define $U = (0,1) \cup (2,3)$. Let $f_n = \chi_{(0,1)} + (-1)^n \chi_{(2,3)}$. This sequence has no limit function, so neither theorem above can apply.*

Our interest in this example is that different hypotheses from the two theorems fail to be satisfied.

Theorem III.21 fails to apply because the domain is not convex, even though the sequence of derivatives $f_n' = 0$ is uniformly convergent.

The Carathéodory version of the theorem fails to apply because the sequence of difference transforms corresponding to any point fails to converge. Take $x = 1/2$ for example. For any $z \in (0,1)$ the difference transform is $\Gamma_n(z) = 0$. For any $z \in (2,3)$ we have

$$f_n(z) - f_n(1/2) = (-1)^n - 1 = \frac{(-1)^n - 1}{z - 1/2}(z - 1/2).$$

So $\Gamma_n(z) \in L(\mathbb{R}, \mathbb{R})$ is multiplication by the fixed real number $\frac{(-1)^n - 1}{z - 1/2}$. This sequence of functions does not converge at any point $z \in (2,3)$.

F. INVERSE FUNCTION THEOREM

One asset of the strong derivative is that it can be used to localize the continuity of the derivative in a way that does not require the derivative to be defined at more than one point. In [34] and [43] the authors use the strong derivative to attain inverse and implicit function theorems for functions which are assumed only to have derivatives at a single point.

We can attain similar results using the Carathéodory condition to change the treatment of the derivative. Here our proof follows the same outline as used in [47], [10], and [43]. Our original goal for this section was a proof as brief as that in [1]. However, this is not possible in infinite dimensional spaces.

Theorem III.24. *Suppose $f : X \to Y$ is strongly Carathéodory differentiable at a point $a \in X$, and the difference transform $\Gamma(a,a) = Df(a)$ is invertible. Then $\exists$ open $U \subseteq X$ and $V \subseteq Y$ such that $a \in U$ and f is 1-1 on U with $f(U) = V$, and f^{-1} is strongly Carathéodory differentiable at $f(a)$ with*

$$Df^{-1}(f(a)) = [Df(a)]^{-1}.$$

Proof. For brevity, define $T = \Gamma(a,a)$. Choose $\lambda \in \mathbb{R}$ so that

$$2\lambda \|T^{-1}\| = 1.$$

The difference transform Γ, guaranteed by strong differentiability, is continuous at (a,a) meaning $\exists$ open $U \subseteq X$ about a such that

$$\Gamma(x,y) \text{ is invertible and } \|\Gamma(x,y) - T\|_{op} < \lambda \ \ \forall x, y \in U.$$

Now to each $y \in Y$ associate the function $h_y : U \to X$ defined by

$$h_y(x) = x + T^{-1}(y - f(x)).$$

Note that $f(x) = y$ if and only if x is a fixed point of h_y. We'll show that h_y is a contraction on U, for every $y \in Y$. To this end, let $x_1, x_2 \in U$ be given.

$$h_y(x_1) - h_y(x_2) = x_1 - x_2 - T^{-1}(f(x_1) - f(x_2))$$

$$x_1 - x_2 - T^{-1}(\Gamma(x_1, x_2)(x_1 - x_2)) = T^{-1}((T - \Gamma(x_1, x_2))(x_1 - x_2)).$$

This implies

$$\begin{aligned} \|h_y(x_1) - h_y(x_2)\|_X &\leq \|T^{-1}\|_{op}\|T - \Gamma(x_1, x_2)\|_{op}\|x_1 - x_2\|_X \\ &< \frac{1}{2\lambda}\lambda\|x_1 - x_2\| = \frac{1}{2}\|x_1 - x_2\|. \end{aligned}$$

Since h_y is a contraction on U, it has at most one fixed point in U. As already noted, this means that for each $y \in Y$ there is at most one $x \in U$ such that $f(x) = y$, i.e. f is 1-1 on U.

To show that f is an open map on U, let open $W \subseteq U$ be given and pick $y_0 = f(x_0) \in f(W)$. We are going to show that y_0 is an interior point of $f(W)$.

Let $r > 0$ be given such that $B = B(x_0; r)$ has the property $\overline{B} \subseteq W$. Let $y \in B(y_0; \lambda r)$ be given. It will be shown that $y \in f(W)$.

$$|h_y(x_0) - x_0| = |T^{-1}(y - y_0)| < \|T^{-1}\|\lambda r = \frac{r}{2}.$$

So if $|x - x_0| \leq r$, then

$$\begin{aligned} |h_y(x) - x_0| &\leq |h_y(x) - h_y(x_0)| + |h_y(x_0) - x_0| \\ &< \frac{1}{2}|x - x_0| + \frac{r}{2} \leq r. \end{aligned}$$

This implies that $h_y(x) \in B$, and $h_y : \overline{B} \to \overline{B}$ is a contraction where $\overline{B} \subseteq X$ is complete. So the Banach contraction mapping theorem implies there is a unique fixed point $x \in \overline{B}$, with $f(x) = y$. So $y \in f(\overline{B}) \subseteq f(W)$ as desired.

To verify the final claims, define $V = f(U)$. Let $c, d \in V$ be given. Suppose $c = f(x)$ and $d = f(z)$. Then

$$f^{-1}(c) - f^{-1}(d) = x - z = [\Gamma(x, z)]^{-1}(f(x) - f(z)).$$

Define $J(c, d) = [\Gamma(f^{-1}(c), f^{-1}(d))]^{-1}$. Note that $J : V \times X \to L(Y, X)$ varies continuously at $(f(a), f(a))$ because f^{-1}, Γ, and Γ^{-1} (by [22, p.584]) are continuous on their domains related to U. This makes J an appropriate difference transform for $f^{-1} : V \to U$ recognizing that

$$(Df^{-1})(f(a)) = J(f(a), f(a)) = [\Gamma(a, a)]^{-1} = [Df(a)]^{-1}.$$

□

And what would a section about the inverse function theorem be without a version of the implicit function theorem? As with the inverse function theorem we find a very good proof in [47].

Theorem III.25. *Let X and Y be Banach spaces. Let $U \subseteq X \times Y$ be open. Suppose that $f : U \to X$ is strongly Carathéodory differentiable, with difference transform Γ, such that $f(a, b) = 0$ for some $(a, b) \in U$. Put $A = \Gamma((a, b), (a, b))$ and assume that A_x (here $A_x : X \to X : h \to A(h, 0)$) is invertible.*

Then there exist open sets $V \subseteq X \times Y$ and $W \subseteq Y$ with $(a, b) \in V$ and $b \in W$ with the following property:

To every $y \in W$ there is a unique x such that

$$(x, y) \in V \text{ and } f(x, y) = 0.$$

Setting $g(y)$ equal to this x gives that

- *g is strongly Carathéodory differentiable from W into X,*
- *$g(b) = a$,*
- *$f(g(y), y) = 0$ for all $y \in W$,*
- *$\Gamma_g(b, b) = -(A_x)^{-1} A_y$. (Here Γ_g is the difference transform for g recognizing item 1.)*

Proof. Define $F : U \to X \times Y$ by $F(x, y) = (f(x, y), y)$. F is a strongly Carathéodory differentiable map from U into $X \times Y$. To see this, note that for all $(x, y), (c, d) \in U$ we have

$$F(x, y) - F(c, d) = (f(x, y), y) - (f(c, d), d) = (\Gamma((x, y), (c, d))((x, y) - (c, d)), I_Y(y - d))$$

where I_Y is the identity on Y. The required continuity comes from the fact that each factor is continuous, and any natural norm we put on $X \times Y$ generates the product topology. We will define Γ_F to be this strong difference transform.

In order to apply Theorem III.24, we need to show that $\Gamma_F((a,b),(a,b))$ is invertible. Note that $\Gamma_F((a,b),(a,b)) = (A, I)$. Suppose that $(A,I)(x,y) = 0$, which is the zero element in $X \times Y$. This assumption gives that $I(y) = 0$, implying that $y = 0$. Then $A(x,y) = A(x,0) = A_x(x)$, which is zero only when $x = 0$ because A_x is invertible.

Now we can apply Theorem III.24 to F at (a,b). We get that there is an open set $V \subseteq X \times Y$ with $(a,b) \in V$ and an open set $N \subseteq X \times Y$ with $(0,b) \in N$ such that $F : V \to N$ is 1-1 and onto.

Let W be the set of all $y \in Y$ such that $(0,y) \in N$. Note that $b \in W$. W is a slice of the open set N, therefore W is open. (Let $y \in W$ be given. Then there is an open box $A \times B$ about $(0,y)$. B is then contained in W and contains y.)

We will now show that the function g exists.

Let $y \in W$ be given. By definition $(0,y) \in N$. Because F is onto N, we have some x such that $(x,y) \in V$ and $F(x,y) = (0,y)$. But then $f(x,y) = 0$ by the definition of F.

Suppose there were some other $z \in X$ such that, with the same y, we had $(z,y) \in V$ and $f(z,y) = 0$. Then

$$F(z,y) = (f(z,y),y) = (0,y) = f((x,y),y) = F(x,y).$$

But F is 1-1, so the function $g : y \to x$ is well-defined.

Let G be the inverse function of F. Consider the following.

$$(g(y) - g(z), y - z) = G(0,y) - G(0,z) = \Gamma_G((0,y),(0,z))(0, y - z).$$

Since G is strongly Carathéodory differentiable (by the Inverse Function Theorem) it follows that the first coordinate above (the function g) is as well.

Finally, to compute $\Gamma_g(b,b)$, let $\Phi(y) = (g(y), y)$. We know this is $G(0,y)$ and differentiable. For all $y \in W$ and all $k \in Y$ we have

$$\Gamma_\Phi(y,y)k = (\Gamma_g(y,y)k, k).$$

We know that $f(\Phi(y)) = 0$ for all $y \in W$. Therefore the chain rule gives

$$\Gamma_f(\Phi(y),\Phi(y)) \circ \Gamma_\Phi(y,y) = 0.$$

When $y = b$ we have $\Phi(y) = (a, b)$ and the outer term is A. So we can re-write

$$A \circ \Gamma_\Phi(y, y) = 0.$$

Defining $A_y(k) = A(0, k)$ and recalling the similar definition of A_x, we can write

$$0 = A \circ \Gamma_\Phi(b, b)k = A \circ (\Gamma_g(b, b)k, k) = A_x\Gamma_g(b, b)k + A_yk$$

for every $k \in Y$. So $A_x\Gamma_g(b, b) + A_y = 0$, implying $\Gamma_g(b, b) = -(A_x)^{-1}A_y$. □

G. FUTURE WORK AND OPEN QUESTIONS

There are numerous applications and generalizations of the inverse function theorem. One might hope that the difference transforms studied in this chapter would facilitate a more succinct proof of the standard inverse function theorem in Banach spaces. This will be pursued further.

In Section 3 of [25], the authors describe a "uniform version" of the inverse function theorem. Their method uses the usual inverse function theorem and is used as a tool to prove their main theorem later in the paper. Their result suggests two questions.

Question III.26. *Can their result be altered using difference transforms?*

Question III.27. *Can these alterations or some other method generate an infinite dimensional version of their result?*

Question III.28. *Is there an earlier use of difference transforms than [9]? Is there an earlier proof of equivalence in Hilbert spaces than [1]?*

Question III.29. *Can difference transforms be used for differential calculus in non-Banach spaces (e.g. L^p with $p < 1$)?*

Question III.16: Exactly which Banach spaces, X, admit difference transforms with off-diagonal continuity? For example, what if X is an Asplund space without an equivalent Frèchet differentiable norm?

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